Section 4.5. Integration and Expectation

Note. In this section we consider integrals of scalar valued functions of a vector and matrix valued functions of a scalar. We touch briefly on the topics of random variables and distribution functions.

Note. For change of variables in the setting of scalar valued functions of a 2-vector or 3-vector, we refer to my online Calculus 3 (MATH 2110) notes on 15.8. Substitution in Multiple Integrals.

Note. Suppose that a region G in the uv-plane is transformed one-to-one into the region R n the xy-plane by equations of the form

$$x = g(u, v),\ \ y = h(u, v).$$

We call R the *image* of G under the transformation, and G the *preimage* of R. Any function $f(x, y)$ defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well. How is the integral of $f(x,)$ over R related to the integral of $f(g(u, v), h(u, v))$ over G? The answer is: If g, h, and f have continuous partials derivatives and $J(u, v)$ is zero only at isolated points, then

$$\int\int_R f(x, y)\, dx\, dy = \int\int_G f(g(u, v), h(u, v))|J(u, v)|\, du\, dv.$$

The factor $J(u, v)$, whose absolute value appears above, is the *Jacobian* of the coordinate transformation. It measures how much the transformation is expanding or contracting the area around a point in G as G is transformed into R.

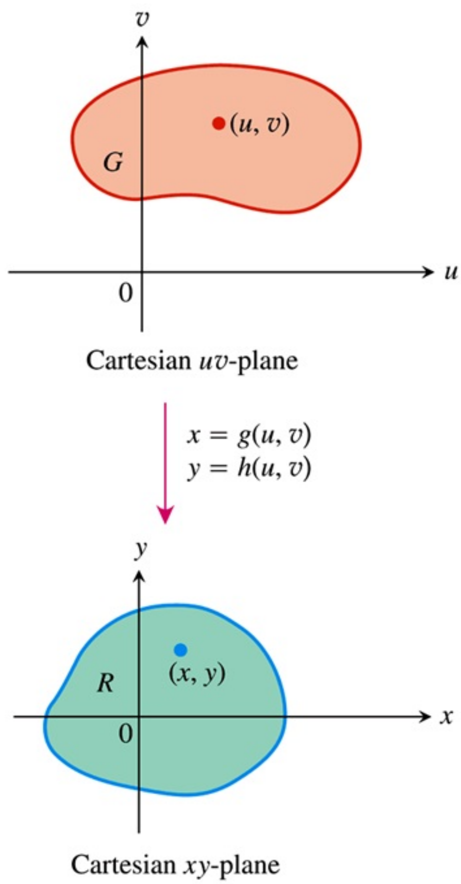

Figure 15.53, Page 905 of *Thomas' Calculus*, 12th edition

Definition. The *Jacobian determinant* or *Jacobian* of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Example. Example 3, page 907. Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2\, dy\, dx$.

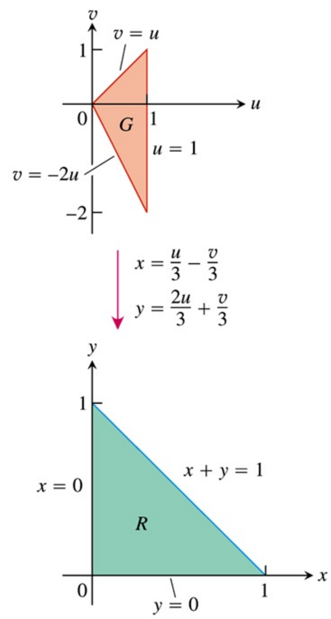

Figure 15.56, Page 907 of *Thomas' Calculus*, 12th edition

Solution. Based on the figure above, we take $x = g(u,v) = u/3 - v/3$ and $y = h(u,v) = 2u/3 + v/3$, but we need to interchange the roles of x and y (since the integral is with respect to y and *then* x), so

$$J(v,u) = \begin{vmatrix} \frac{\partial y}{\partial v} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial x}{\partial u} \end{vmatrix} = \begin{vmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{vmatrix} = \frac{1}{3}.$$

So

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2\, dy\, dx = \int\int_G f(g(u,v),h(u,v))|J(v,u)|\, dv\, du$$

$$= \int_0^1 \int_{-2u}^{2u} \sqrt{\left(\frac{u}{3}-\frac{v}{3}\right)+\left(\frac{2u}{3}+\frac{v}{3}\right)} \left(\left(\frac{2u}{3}+\frac{v}{3}\right) - 2\left(\frac{u}{3}-\frac{v}{3}\right)\right)^2 \left|\frac{1}{3}\right| dv\, du$$

$$= \frac{1}{3}\int_0^1 \int_{-2u}^{u} \sqrt{u}v^2\, dv\, du = \frac{1}{3}\int_0^1 \left(\left(\frac{\sqrt{u}v^3}{3}\right)\Bigg|_{v=-2u}^{v=u}\right) du$$

$$= \frac{1}{9} \int_0^1 (\sqrt{u} u^3 + \sqrt{u} 8u^3)\, du = \int_0^1 u^{7/2}\, du = \frac{2}{9}\, u^{9/2}\Big|_0^1 = \frac{2}{9}.$$

Note. Suppose that a region G in uvw-space is transformed one-to-one into the region D in xyz-space by differentiable equations of the form

$$x = g(u, v, w),\ \ y = h(u, v, w),\ \ z = k(u, v, w).$$

Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

defined on G. If g, h, and k have continuous first partial derivatives, then the integral of $F(x, y, z)$ over D is related to the integral of $H(u, v, w)$ over G by the equation

$$\int\int\int_D F(x, y, z)\, dx\, dy\, dz = \int\int\int_G H(u, v, w)|J(u, v, w)|\, du\, dv\, dw.$$

The factor $J(u, v, w)$ whose absolute value appears in this equation, is the *Jacobian determinant*

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This determinant measures how much the volume near a point in G is being expanded or contracted by the transformation from (u, v, w) to (x, y, z) coordinates.

Note. Just as we differentiated a matrix function of a single variable, $A(x) = [a_{ij}(x)]$, we can also integrate $A(x)$ for $x \in [a, b]$ to produce matrix

$$\int_a^b A(x)\, dx = \left[\int_a^b a_{ij}(x)\, dx\right].$$

Here and throughout, we assume that all functions are integrable over the domain of integration set.

Theorem 4.5.1. For $A(x) = [a_{ij}(x)]$ an $n \times n$ matrix function of scalar variable x, we have

$$\int_a^b \text{tr}(A(x))\,dx = \text{tr}\left(\int_a^b A(x)\,dx\right).$$

Note. Gentle claims the following result is a consequence of the Lebesgue Dominated Convergence Theorem (see page 3 of my notes from Real Analysis 1 (MATH 5210) on 4.4. The General Lebesgue Integral and any standard text on real analysis, though it is in neither Royden and Fitzpatrick's *Real Analysis*, 4th Edition [which we use in Real Analysis 1 and 2, MATH 5210/5220] nor Walter Rudin's *Real & Complex Analysis*, 3rd Edition). But it is nice to see this reference to a measure theoretic result!

Theorem 4.5.2. Let $\mathcal{X}$ be an open set in $\mathbb{R}^n$ and let $f(x,y)$ and $\partial f/\partial x$ be scalar-valued functions that are continuous on $\mathcal{X} \times \mathcal{Y}$ for some set $\mathcal{Y}$ in $\mathbb{R}^n$. Suppose there are scalar functions $g_0(y)$ and $g_1(y)$ such that

$$\left.\begin{array}{c} |f(x,y)| \le g_0(y) \\ \left\|\frac{\partial}{\partial x} f(x,y)\right\| \end{array}\right\} \text{ for all } (x,y) \in \mathcal{X} \times \mathcal{Y},$$

$$\int_{\mathcal{Y}} g_0(y)\,dy < \infty \text{ and } \int_{\mathcal{Y}} g_1(y)\,dy < \infty.$$

Then

$$\frac{\partial}{\partial x}\int_{\mathcal{Y}} f(x,y)\, fy = \int_{\mathcal{Y}} \frac{\partial}{\partial x} f(x,y)\, dy.$$

Definition. A *vector random variable* is a function from some sample space X into $\mathbb{R}^n$. A *matrix random variable* is a function from a sample space into $\mathbb{R}^{n\times m}$. A *distribution function* is associated with each random variable which integrates to 1 over the whole sample space.

Definition. The *d-variate normal distribution* is

$$f(x) = \frac{1}{(2\pi)^{d/2}(\det(\Sigma))^{1/2}} e^{-(x-\mu)^T\Sigma^{-1}(x-\mu)/2}$$

where Σ^{-1} is a symmetric positive definite $d\times d$ matris, μ is a constant d-vector, and $x\in\mathbb{R}^d$.

Note. We expect the integral of the d-variate dimension over all of $\mathbb{R}^d$ to be 1. The following result establishes this.

Theorem 4.5.3. Atiken's Integral.

For Σ^{-1} a symmetric positive definite $d\times d$ matrix, is a constant d-vector, and $x\in\mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} e^{-(x-\mu)^T\Sigma^{-1}(x-\mu)/2}\, dx = (2\pi)^{d/2}(\det(\Sigma))^{1/2}.$$

Definition. The *expected value* of function f of a vector-valued random variable X where X ranges over domain $D(X)$ is

$$E(f(X)) = \int_{D(X)} f(x)p_X(x)\,dx$$

where $p_X(x)$ is the probability density function.

Theorem 4.5.4. The expected value of $f(x) = x$ with respect to the d-variate normal distribution

$$p_X(x) = (2\pi)^{-d/2}(\det(\Sigma))^{1/2}e^{-(x-\mu)^T\Sigma^{-1}(x-\mu)/2}$$

is μ. (Notice that $X = \mathbb{R}^d$ here.)

Note. A proof of Theorem 4.5.4 is to be given in Exercise 4.5.A.

Definition. The *variance* of vector valued random variable X is the matrix

$$V(X) = E((X - E(X))(X - E(X)^T).$$

Theorem 4.5.5. The variance of X with respect to the d-variate normal distribution is Σ.

Note. A proof of Theorem 4.5.5 is to be given in Exercise 4.5.B.