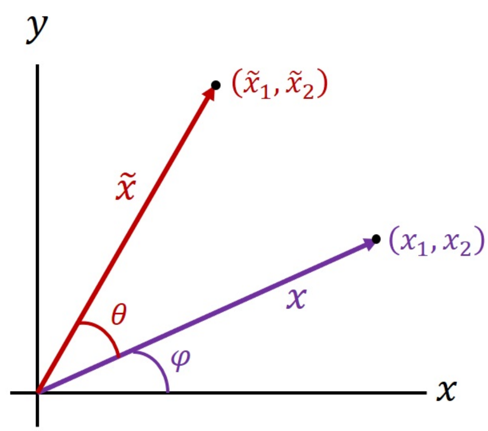
Section 5.2. Geometric Transformations

Note. Gentle states that "... a vector represents a point in space...." This is misleading. We can associate a vector in $\mathbb{R}^n$, $v = (x_1, x_2, \ldots, x_n)$, with a point in $\mathbb{R}^n$, $p = (x_1, x_2, \ldots, x_n)$ by representing the vector as an arrow in *standard position* with its tail at the origin and its head at point p. Often, vectors are notationally distinguished from points by using square brackets for vectors and parentheses for points: vector $v = [x_1, x_2, \ldots, x_n]$ and point $p = (x_1, x_2, \ldots, x_n)$; see my online notes for Linear Algebra (MATH 2010) on 1.1. Vectors in Euclidean Spaces. Vectors and points in $\mathbb{R}^n$ are very different. For example, vectors can be added together and multiplied by scalars, but vectors don't have a location. Points cannot be added together nor multiplied by scalars, but points do have a location.

Definition. A transformation that preserves lengths and angles is an *isometric transformation*. A transformation that preserves angles is an *isotropic transformation*; a transformation that does not preserve angles is *anisotropic*. A transformation of the form mapping x to $x+t$ (where $x, t \in \mathbb{R}^n$) is a *translation transformation*.

Note. Gentle states that all transformations in this section "are linear transformations because they preserve straight lines" (page 175). This is unusual and a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is usually defined as satisfying $T(ax + by) = aT(x) + bT(y)$ for all $a, b \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. In this case, $T(0) = 0$ (since $T(0) = T(0 + 0) = T(0) + T(0)$), so a translation is not an example of a linear transformation in this sense. You see in Linear Algebra that all linear transformations (in this traditional sense) from $\mathbb{R}^m$ to $\mathbb{R}^n$ are represented by $n \times m$ matrices; see my online Linear Algebra (MATH 2010) notes on 2.3. Linear Transformations of Euclidean Spaces.

Note. Consider a vector x in $\mathbb{R}^2$ with components x_1 and x_2. In standard position in the $\mathbb{R}^2$ ("geometric") Cartesian plane, we can represent x as an arrow from the origin to the point (x_1, x_2). We now find a transformation that rotates x about the origin through an angle θ. With φ as the angle between the positive x-axis and vector x we have $x_1 = \|x\| \cos\varphi$ and $x_2 = \|x\| \sin\varphi$. With x rotated through angle θ we produce $\tilde{x}$ with endpoints at the origin and the point $(\tilde{x}_1, \tilde{x}_2)$.



Then the angle between the positive x-axis and $\tilde{x}$ is $\varphi+\theta$ and so $\tilde{x}_1 = \|x\| \cos(\varphi+\theta)$ and $\tilde{x}_2 = \|x\| \sin(\varphi+\theta)$. The summation formulae for sine and cosine are

$$\begin{aligned} \cos(\varphi+\theta) &= \cos\varphi\cos\theta - \sin\varphi\sin\theta \\ \sin(\varphi+\theta) &= \sin\varphi\cos\theta + \cos\varphi\sin\theta. \end{aligned}$$

Now $\cos\varphi = x_1/\|x\|$ and $\sin\varphi = x_2/\|x\|$, so

$$\begin{aligned} \tilde{x}_1 &= \|x\|((x_1/\|x\|)\cos\theta - (x_2/\|x\|)\sin\theta) = x_1\cos\theta - x_2\sin\theta \\ \tilde{x}_2 &= \|x\|((x_2/\|x\|)\cos\theta + (x_1/\|x\|)\sin\theta) = x_2\cos\theta + x_1\sin\theta. \end{aligned}$$

So with $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ we have $\tilde{x} = Ax$. Notice that A is an orthogonal matrix.

Note. In $\mathbb{R}^3$, to rotate a 3-vector about the y-axis we use

$$B = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

The reason for the signs of the sines (!) is that in a right hand coordinate system if we rotate the xz-plane into the xy-plane as given in the image above then the y-axis points down into the image. Then a positive angle in the image represents a negative rotation about the y-axis (so to generate B we add a middle row and column to A with entries 0, 1, 0 [in this order] and then replace θ with $-\theta$ in the trig functions). Similarly, to rotate a 3-vector about the x-axis we use

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

Of course a rotation about the z-axis in $\mathbb{R}^3$ results from

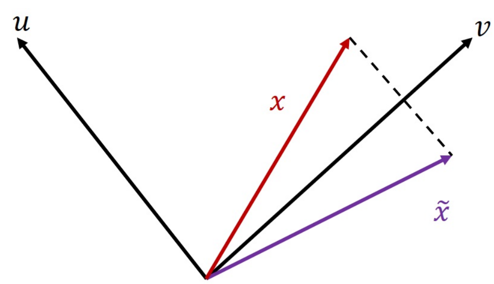
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Euler's Rotation Theorem implies that a rotation in $\mathbb{R}^3$ about any axis (through the origin) is a combination of rotations about the x, y, and z axis. In fact, the collection of all rotation matrices form an infinite group called the "3D rotation group" or the " group of special orthogonal 3×3 matrices" denoted $SO(3)$. The elements of $SO(3)$ are all orthogonal 3×3 matrices of determinant 1. The group of all 3×3 orthogonal matrices form the "orthogonal group" $O(3)$.

Note. In sophomore Linear Algebra (MATH 2010) you encounter reflections in $\mathbb{R}^2$ about the x-axis, y-axis, and the line $y = x$ (see my online Linear Algebra notes on 2.4. Linear Transformations of the Plane). This idea can be generalized to reflections about any line in any direction using unit vectors.

Definition. Let u and v for orthonormal vectors and let x be a vector in the space spanned by u and v (in which case we can represent the span of u and v as a two-dimensional plane). Then $x = c_1u + c_2v$ for some scalars c_1 and c_2. The vector $\tilde{x} = -c_1u + c_2v$ is the *reflection* of x about v in the direction u.

Note. Geometrically, the reflection of x about v in the direction u can be represented as:



Note. As commented above, translations are not (in the usual sense) linear transformations and so cannot be represented by matrix multiplication. We introduce a new set of coordinates which allows us to "translate" vectors into the new coordinates and then represent translation by matrix multiplication in the new coordinates.

Definition. For point $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ we introduce *homogeneous coordinates* $(x_0^h, x_1^h, x_2^h, \ldots, x_d^h) \in \mathbb{R}^{d+1}$ where $x_1^h = x_0^h x_1$, $x_2^h = x_0^h x_2$, $\ldots$, $x_d^h = x_0^h x_d$.

Note. Homogeneous coordinates in $\mathbb{R}^{d+1}$ *represent* points in $\mathbb{R}^d$. When $x_0^h = 1$, we have $x_i^h = x_i$ for $1 \le i \le d$. When $x_0^h = 0$ the homogeneous coordinate corresponds to no point in $\mathbb{R}^d$; Gentle says that the collection of all homogeneous coordinate elements of $\mathbb{R}^{d+1}$ where $x_0^h = 0$ corresponds in projective geometry to the "hyperplane at infinity" (page 179).

Note. We now represent the translation transformation mapping x to $x + t = \tilde{x}$, where $x, t \in \mathbb{R}^d$, $x = [x_1, x_2, \ldots, x_d]^T$, and $t = [t_1, t_2, \ldots, t_d]^T$, with matrix multiplication and homogeneous coordinates. First we represent the point $(x_1, x_2, \ldots, x_d)$ in homogeneous coordinates as $(1, x_1, x_2, \ldots, x_d)$. Then for the $(d+1) \times (d+1)$ matrix

$$T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ t_d & 0 & \cdots & 1 \end{bmatrix},$$

the vector $x^h = [1, x_1, x_2, \ldots, x_d]^T$ satisfies $Tx^h = [1, x_1 + t_1, x_2 + t_2, \ldots, x_d + t_d]^T = \tilde{x}^h$. Now $\tilde{x}^h$ is "associated" with the point $(1, x_1 + t_1, x_2 + t_2, \ldots, x_d + t_d)$ which is the homogeneous coordinates associated with the vector $x + t = \tilde{x} \in \mathbb{R}^d$. Therefore, translation by any vector $t \in \mathbb{R}^d$ is represented by matrix multiplication by T, but we must first convert/associate x with homogeneous coordinates, then perform the matrix multiplication, and finally convert/associate the homogeneous coordinates with the vector $x + t$ in $\mathbb{R}^d$. As Gentle insightfully states: "We must be careful to distinguish the point x from the vector that represents the point" (page 179).