Section 5.2. Geometric Transformations

Note. Gentle states that “...a vector represents a point in space...” This is misleading. We can associate a vector in \( \mathbb{R}^n \), \( v = (x_1, x_2, \ldots, x_n) \), with a point in \( \mathbb{R}^n \), \( p = (x_1, x_2, \ldots, x_n) \) by representing the vector as an arrow in standard position with its tail at the origin and its head at point \( p \). Often, vector are notationally distinguished from vectors by using square brackets for vectors and parentheses for points: vector \( v = [x_1, x_2, \ldots, x_n] \) and point \( p = (x_1, x_2, \ldots, x_n) \); see my notes from Linear Algebra (MATH 2010): http://faculty.etsu.edu/gardnerr/2010/c1s1.pdf.

Vectors and points in \( \mathbb{R}^n \) are very different. For example, vectors can be added together and multiplied by scalars, but vectors don’t have a location. Points cannot be added together nor multiplied by scalars, but points do have a location.

Definition. A transformation that preserves lengths and angles is an isometric transformation. A transformation that preserves angles is an isotropic transformation; a transformation that does not preserve angles is anisotropic. A transformation of the form mapping \( x \) to \( x + t \) (where \( x, t \in \mathbb{R}^n \)) is a translation transformation.

Note. Gentle states that all transformations in this section “are linear transformations because they preserve straight lines” (page 175). This is unusual and a linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is usually defined as satisfying \( T(ax + by) = aT(x) + bT(y) \) for all \( a, b \in \mathbb{R} \) and \( x, y \in \mathbb{R}^n \). In this case, \( T(0) = 0 \) (since \( T(0) = T(0 + 0) = T(0) + T(0) \)), so a translation is not an example of a linear transformation in this sense. You see in Linear Algebra that all linear transformations (in this traditional sense) form \( \mathbb{R}^n \) to \( \mathbb{R}^m \) are represented by an \( n \times m \) matrix; see http://faculty.etsu.edu/gardnerr/2010/c2s3.pdf.
Note. Consider a vector $x$ in $\mathbb{R}^2$ with entries $x_1$ and $x_2$. In standard position in the $\mathbb{R}^2$ (“geometric”) Cartesian plane we can represent $x$ as an arrow from the origin to the point $(x_1, x_2)$. We now find a transformation that rotates $x$ about the origin through an angle $\theta$. With $\phi$ as the angle between the positive $x$-axis and vector $x$ we have $x_1 = ||x|| \cos \varphi$ and $x_2 = ||x|| \sin \varphi$. With $x$ rotated through angle $\theta$ we produce $\tilde{x}$ with endpoints at the origin and the point $(\tilde{x}_1, \tilde{x}_2)$.

Then the angle between the positive $x$-axis and $\tilde{x}$ is $\varphi + \theta$ and so $\tilde{x}_1 = ||x|| \cos(\varphi + \theta)$ and $\tilde{x}_2 = ||x|| \sin(\varphi + \theta)$. Now the summation formulae for sine and cosine are

\[
\cos(\varphi + \theta) = \cos \varphi \cos \theta - \sin \varphi \sin \theta
\]
\[
\sin(\varphi + \theta) = \sin \varphi \cos \theta + \cos \varphi \sin \theta.
\]

Now $\cos \varphi = x_1/||x||$ and $\sin \varphi = x_2/||x||$, so

\[
\tilde{x}_1 = ||x||((x_1/||x||) \cos \theta - (x_2/||x||) \sin \theta) = x_1 \cos \theta - x_2 \sin \theta
\]
\[
\tilde{x}_2 = ||x||((x_2/||x||) \cos \theta + (x_1/||x||) \sin \theta) = x_2 \cos \theta + x_1 \sin \theta.
\]

So with $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ we have $\tilde{x} = Ax$. Notice that $A$ is an orthogonal matrix.
Note. In $\mathbb{R}^3$, to rotate a 3-vector about the $y$-axis we use

$$B = \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}.$$ 

The reason for the signs of the sines (!) is that in a right hand coordinate system if we rotate the $xz$-plane into the $xy$-plane as given in the image above then the $y$-axis points down into the image. Then a positive angle in the image represents a negative rotation about the $y$-axis (so to generate $B$ we ass a middle row and column to $A$ with entries $0, 1, 0$ [in this order] and then replace $\theta$ with $-\theta$ in the trig functions). Similarly, to rotate a 3-vector about the $x$-axis we use

$$C = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}.$$ 

Of course a rotation about the $z$-axis in $\mathbb{R}^3$ results from

$$\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}.$$ 

Euler’s Rotation Theorem implies that a rotation in $\mathbb{R}^3$ about any axis (through the origin) is a combination of rotations about the $x$, $y$, and $z$ axis. In fact, the collection of all rotation matrices form an infinite group called the “3D rotation group” or the “group of special orthogonal 3 $\times$ 3 matrices” denoted $SO(3)$. The elements of $SO(3)$ are all orthogonal 3 $\times$ 3 matrices of determinant 1. The group of all 3 $\times$ 3 orthogonal matrices form the “orthogonal group” $O(3)$. 

Note. In sophomore Linear Algebra (MATH 2010) you encounter reflections in $\mathbb{R}^2$ about the $x$-axis, $y$-axis, and the line $y = x$ (see http://faculty.etsu.edu/gardnerr/2010/c2s4.pdf). This idea can be generalized to reflections about any line in any direction using unit vectors.

**Definition.** Let $u$ and $v$ for orthonormal vectors and let $x$ be a vector in the space spanned by $u$ and $v$ (in which case we can represent the space of $u$ and $v$ as a two-dimensional plane). Then $x = c_1u + c_2v$ for some scalars $c_1$ and $c_2$. The vector $\tilde{x} = -c_1u + c_2v$ is the *reflection* of $x$ about $v$ in the direction $n$.

Note. Geometrically, the reflection of $x$ about $u$ in the direction $v$ can be represented as:

![Diagram](attachment:image.png)

Note. As commented above, translations are not (in the usual sense) linear transformations and so cannot be represented by matrix multiplication. We introduce a new set of coordinates which allows us to “translate” vectors into the new coordinates and then represent translation by matrix multiplication in the new coordinates.
5.2. Geometric Transformations

**Definition.** For point \((x_1, x_2, \ldots, x_d) \in \mathbb{R}^d\) we introduce *homogeneous coordinates* 
\((x_h^0, x_h^1, x_h^2, \ldots, x_h^d) \in \mathbb{R}^{d+1}\) where

\[x_1^h = x_0^h x_1, \quad x_2^h = x_0^h x_2, \quad \ldots, \quad x_d^h = x_0^h x_d.\]

**Note.** Homogeneous coordinates in \(\mathbb{R}^{d+1}\) represent points in \(\mathbb{R}^d\). When \(x_0^h = 1\), we have \(x_i^h = x_i\) for \(1 \leq i \leq d\). When \(x_0^h = 0\) the homogeneous coordinate corresponds to no point in \(\mathbb{R}^d\); Gentle says that the collection of all homogeneous coordinate elements of \(\mathbb{R}^{d+1}\) where \(x_0^h = 0\) corresponds in projective geometry to the “hyperplane at infinity” (page 179).

**Note.** We now represent the translation transformation mapping \(x\) to \(x + t = \tilde{x}\), where \(x, t \in \mathbb{R}^d\), \(x = [1, x_2, \ldots, x_d]^T\), and \(t = [t_1, t_2, \ldots, t_d]^T\), with matrix multiplication and homogeneous coordinates. First we represent the point \((x_1, x_2, \ldots, x_d)\) in homogeneous coordinates as \((1, x_2, x_2, \ldots, x_d)\). Then for the \((d + 1) \times (d + 1)\) matrix

\[
T = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
t_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
t_d & 0 & \cdots & 1
\end{bmatrix},
\]

the vector \(x^h = [1, x_1, x_2, \ldots, x_d]^T\) satisfies \(T x^h = [1, x_1 + t_1, x_2 + t_2, \ldots, x_d + t_d]^T = \tilde{x}^h\). Now \(\tilde{x}^h\) is “associated” with the point \((a, x_1 + t_1, x_2 + t_2, \ldots, x_d + t_d)\) which is the homogeneous coordinates associated with the vector \(x + t = \tilde{x} \in \mathbb{R}^d\). Therefore, translation by any vector \(t \in \mathbb{R}^d\) is represented by matrix multiplication by \(T\), but we must first convert/associate \(x\) with homogeneous coordinates, then perform the
matrix multiplication, and finally convert/associate the homogeneous coordinates with the vector $x + t$ in $\mathbb{R}^d$. As Gentle insightfully states: “We must be careful to distinguish the point $x$ from the vector that represents the point” (page 179).