Section 5.3. Householder Transformations (Reflections)

Note. In the previous section we introduced a technique to reflect a vector x about a vector v in the direction u (where x is in the span of $\{u, v\}$). In this section, we accomplish such a reflection using matrix multiplication.

Note. Let u and v be orthonormal vectors and let x be in the span of $\{u, v\}$ so that $x = c_1 u + c_2 v$. Recall that the reflection of x about v in the direction u is $\tilde{x} = -c_1 u + c_2 v$. Consider the matrix $H = I - 2uu^T$ (H is $n \times n$ provided u, v, and x are in $\mathbb{R}^n$). Notice

$$\begin{aligned} Hx &= (I - 2uu^T)(c_1 u + c_2 v) = c_1 u + c_2 v - 2c_1 uu^T u - 2c_2 uu^T v \\ &= c_1 u + c_2 v - 2c_1 u\langle u, u\rangle - 2c_2 u \langle u, v\rangle \\ &= c_1 u + c_2 v - 2c_1 u \text{ since } u \text{ and } v \text{ are orthonormal} \\ &= -c_1 u + c_2 v = \tilde{x}. \end{aligned}$$

So we can represent reflection about v in the direction u (for x in span$\{u, v\}$) with matrix multiplication by matrix H.

Definition. For orthonormal u and v, the matrix $H = I - 2uu^T$ is a *reflection matrix* or a *Householder matrix*.

Note. We immediately have that $H = I - 2uu^T$ satisfies:

(1) $Hu = -u$

(2) $Hv = v$

(3) $H = H^T$ (H is symmetric)

(4) $H^T = H = H^{-1}$.

Theorem 5.3.1. For nonzero $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$, define

$$q = [x_1, x_2, \ldots, x_{k-1}, x_k + \mathrm{sgn}(x_k)\|x\|_2, x_{k+1}, \ldots, x_n]^T$$

and let $u = q/\|q\|_2$. Then $H = I - 2uu^T$ maps x to $Hx = \tilde{x}$ where all entries of $\tilde{x}$ are 0 except for the kth entry (when $x_k \neq 0$).

Note. We see in the proof of Theorem 5.3.1 that the kth entry of $\tilde{x} = Hx$ is $-\mathrm{sgn}(x_k)\|x\|_2$. Gentle gives some of the computations to illustrate the process for $x = [3, 1, 2, 1, 1]^T$ and $k = 1$ (see page 181). Notice that $\|x\|_2 = 4$. We have $Hx = \tilde{x} = [-4, 0, 0, 0, 0]^T$, as expected.

Revised: 7/4/2020