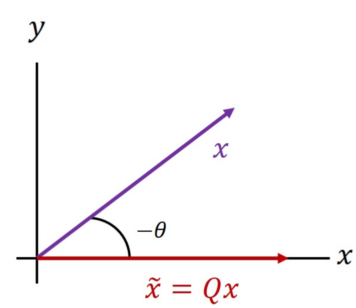
Section 5.4. Givens Transformations (Rotations)

Note. We consider a transformation that leaves all but two entries of a vector fixed and maps one of the remaining entries to 0 (the Givens transformation). We then apply the matrix representing this transformation to matrices in such a way as to leave all but two rows and columns fixed and to map a diagonal entry to a given value.

Note. In $\mathbb{R}^2$, we can use a rotation to map $x = [x_1, x_2]^T$ to $\tilde{x} = [\tilde{x}_1, 0]^T$ where $\tilde{x}_1 = \|x\|$. We simply need $\cos(-\theta) = x_1/\|x\|$ and $\sin(-\theta) = x_2/\|x\|$ in the rotation represented by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ of Section 5.2. That is, we consider

$$\begin{bmatrix} x_1/\|x\| & x_2/\|x\| \\ -x_2/\|x\| & x_1/\|x\| \end{bmatrix} = \frac{1}{\|x\|}\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}:$$



Then

$$Qx = \frac{1}{\|x\|}\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|x\| \\ 0 \end{bmatrix} = \tilde{x}.$$

Notice that we could perform a similar rotation of a plane, even if the plane was embedded in a higher dimensional space; if there was a z-axis added to the figure above, for example, the z component of a rotated vector would remain the same in the rotation while the y component goes to 0 and the x component is modified.

Note. We now generalize the above idea by mapping $x = [x_1, x_2, \ldots, x_p, \ldots, x_q, \ldots x_n]^T$ to $\tilde{x} = [x_1, x_2, \ldots, x_{p-1}, \tilde{x}_p, x_{p+1}, \ldots x_{q-1}, 0, x_{q+1}, \ldots x_n]^T$. So $\tilde{x}$ has the same entries as x, except that the pth entry of $\tilde{x}$ is $\tilde{x}_p = \sqrt{x_p^2 + x_q^2}$, and the qth entry of $\tilde{x}$ is 0. With $n = 2$, $p = 1$, and $q = 2$ this reduces to the example above. Think of this mapping as a rotation of the (x_p, x_q)-plane that leaves the perp space of this plane fixed (just as the rotation of the xy-plane above can be thought of as leaving the z-axis fixed).

Definition. A transformation mapping $x = [x_1, x_2, \ldots, x_p, \ldots, x_q, \ldots x_n]^T$ to

$$\tilde{x} = [x_1, x_2, \ldots, x_{p-1}, \tilde{x}_p, x_{p+1}, \ldots x_{q-1}, 0, x_{q+1}, \ldots x_n]^T$$

(as above) is a *Givens transformation.*

Theorem 5.4.1. A matrix which produces the Givens Transformation is

$$G_{pq} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & c & 0 & \cdots & 0 & s & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -s & 0 & \cdots & 0 & c & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where $c = x_p/\sqrt{x_p^2+x_q^2}$, $s = x_q/\sqrt{x_p^2+x_q^2}$, $g_{pp} = g_{qq} = c$, $g_{pq} = -g_{qp} = s$, and $\tilde{x}_p = cx_p + sx_q = (x_p^2+x_q^2)/\sqrt{x_p^2+x_q^2} = \sqrt{x_p^2+x_q^2}$.

Proof. With $x = [x_1, x_2, \ldots, x_p, \ldots, x_q, \ldots, x_n]^T$ we have

$$G_{pq}x = [x_1, x_2, \ldots, x_{p-1}, cx_p + sx_q, x_{p+1}, \ldots, x_{q-1}, -sx_p + cx_q, x_{q+1}, \ldots, x_n]^T$$

$$= [x_1, x_2, \ldots, x_{p-1}, \tilde{x}_p, x_{p+1}, \ldots x_{q-1}, 0, x_{q+1}, \ldots x_n]^T = \tilde{x},$$

as claimed. ∎

Note. More generally, we can use the matrix G_{pq} representing the Givens transformation to transform a symmetric matrix X into a symmetric matrix $\tilde{X}$ with the same rows and columns as X, except for the pth and qth rows and columns. We require the (p,p) entry of $\tilde{X}$ be some given value, say $\tilde{x}_{pp} = a$. We want a matrix G_{pq} as given in Theorem 5.2.1 such that $\tilde{X} = G_{pq}^T X G_{pq}$. We need only consider $\tilde{x}_{pp}$, since G_{pq} and G_{pq}^T will preserve all rows and columns of X, except possibly for the pth and qth rows and columns. So we need

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix}^T \begin{bmatrix} x_{pp} & x_{pq} \\ x_{pq} & x_{qq} \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} a & \tilde{x}_{pq} \\ \tilde{x}_{pq} & \tilde{x}_{qq} \end{bmatrix}$$

where $c^2 + s^2 = 1$. Now

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x_{pp} & x_{pq} \\ x_{pq} & x_{qq} \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} cx_{pp} - sx_{pq} & cx_{pq} - sx_{qq} \\ sx_{pp} + cx_{pq} & sx_{pq} + cx_{qq} \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$\begin{bmatrix} c(cx_{pp} - sx_{pq}) - s(cx_{pq} - sx_{qq}) & s(cx_{pp} - sx_{pq}) + c(cx_{pq} - sx_{qq}) \\ c(sx_{pp} + cx_{pq}) - s(sx_{pq} + cx_{qq}) & s(sx_{pp} + cx_{pq}) + c(sx_{pq} + cx_{qq}) \end{bmatrix}$$

and so we need

$$a = c^2 x_{pp} - csx_{pq} - csx_{pq} + s^2 x_{qq} = c^2 x_{pp} - 2csx_{pq} + s^2 x_{qq}.$$

Now

$$\begin{aligned} a = c^2 x_{pp} - 2cs x_{pq} + s^2 x_{qq} &= c^2(x_{pp} - 2(s/c)x_{pq} + (s/c)^2 x_{qq}) \\ &= c^2(x_{pp} - 2t x_{pq} + t^2 x_{qq}) \text{ where } t = s/c \\ &= \frac{1}{1+t^2}(x_{pp} - 2t x_{pq} + t^2 x_{qq}) \text{ since } 1 + t^2 = 1 + s^2/c^2 \\ &\quad = (c^2 + s^2)/c^2 = 1/c^2 \text{ and so } 1/(1+t^2) = c^2, \end{aligned}$$

and so we need

$$x_{pp} - 2t x_{pq} + t^2 x_{qq} = (1 + t^2)a$$

or

$$(x_{qq} - a)t^2 - 2x_{pq}t + (x_{pp} - a) = 0.$$

The values of t satisfying this equation are

$$t = \frac{2x_{pq} \pm \sqrt{4x_{pq}^2 - 4(x_{qq} - a)(x_{pp} - a)}}{2(x_{qq} - a)} = \frac{x_{pq} \pm \sqrt{x_{pq}^2 - (x_{qq} - a)(x_{pp} - a)}}{(x_{qq} - a)},$$

and there are real solutions if and only if

$$x_{pq}^2 \geq (x_{pp} - a)(x_{qq} - a).$$

If this condition holds, then choose the value for t of

$$t = \frac{x_{pq} + \sqrt{x_{pq}^2 - (x_{qq} - a)(x_{pp} - a)}}{(x_{qq} - a)}.$$

Next, let $c = 1/\sqrt{1 + t^2}$ and, since $t = s/c$, let $s = ct$. Then G_{pq} with these values of c and s yield the desired transformation. We summarize this in a theorem.

Theorem 5.4.2. Let X be a symmetric matrix and $1 \leq p < q \leq n$. For given $a \in \mathbb{R}$, if $x_{pq}^2 \geq (x_{qq} - a)(x_{pp} - a)$ then with $c = 1/\sqrt{1+t^2}$ and $s = ct$, where

$$t = \frac{x_{pq} + \sqrt{x_{pq}^2 - (x_{qq} - a)(x_{pp} - a)}}{(x_{qq} - a)},$$

we have $\tilde{X} = G_{pq}^T X G_{pq}$ (where G_{pq} is the matrix from Theorem 5.4.1) is a symmetric matrix with the same rows and columns as X, except for the pth and qth rows and columns, and $\tilde{x}_{pp} = a$.

Revised: 7/11/2020