Section 5.4. Givens Transformations (Rotations)

Note. We consider a transformation that leaves all but two entries of a vector fixed and maps one of the remaining entries to 0 (the Givens transformation). We then apply the matrix representing this transformation to matrices in such a way as to leave all but two rows and columns fixed and to map a diagonal entry to a given value.

Note. In \( \mathbb{R}^2 \), we can use a rotation to map \( x = [x_1, x_2]^T \) to \( \tilde{x} = [\tilde{x}_1, 0]^T \) where \( \tilde{x}_1 = \|x\| \). We simply need \( \cos(-\theta) = x_1/\|x\| \) and \( \sin(-\theta) = x_2/\|x\| \) in the rotation represented by
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]
of Section 5.2. That is, we consider
\[
\begin{pmatrix}
x_1/\|x\| & x_2/\|x\| \\
-x_2/\|x\| & x_1/\|x\|
\end{pmatrix}
= \frac{1}{\|x\|}
\begin{pmatrix}
x_1 & x_2 \\
-x_2 & x_1
\end{pmatrix}:
\]

Then
\[
QX = \frac{1}{\|x\|}
\begin{pmatrix}
x_1 & x_2 \\
-x_2 & x_1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix}
\|x\| \\
0
\end{pmatrix} = \tilde{x}.
\]
Note. We now generalize the above idea by mapping \( x = [x_1, x_2, \ldots, x_p, \ldots, x_q, \ldots x_n]^T \)
to
\[
\tilde{x} = [a_1, x_2, \ldots, x_{p-1}, \tilde{x}_p, x_{p+1}, \ldots x_{q-1}, 0, x_{q+1}, \ldots x_n]^T.
\]
So \( \tilde{x} \) has the same entries as \( x \), except that the \( p \)th entry of \( \tilde{x} \) is \( \tilde{x}_p = \sqrt{x_p^2 + x_q^2} \), and the \( q \)th entry of \( \tilde{x} \) is 0. With \( n = 1 \), \( p = 1 \), and \( q = 2 \) this reduces to the example above.

Definition. A transformation mapping \( x = [x_1, x_2, \ldots, x_p, \ldots, x_q, \ldots x_n]^T \) to
\[
\tilde{x} = [a_1, x_2, \ldots, x_{p-1}, \tilde{x}_p, x_{p+1}, \ldots x_{q-1}, 0, x_{q+1}, \ldots x_n]^T
\]
(as above) is a Givens transformation.

Theorem 5.4.1. A matrix which produces the Givens Transformation is
\[
G_{pq} = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & c & 0 & \cdots & s & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -s & 0 & \cdots & c & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
\end{bmatrix},
\]
where \( c = \frac{x_p}{\sqrt{x_p^2 + x_q^2}}, \ s = \frac{x_q}{\sqrt{x_p^2 + x_q^2}}, \ g_{pp} = g_{qq} = c, \ g_{pq} = -g_{qp} = s, \) and \( \tilde{x}_p = cx_p + sx_q = (x_p^2 + x_q^2) / \sqrt{x_p^2 + x_q^2} = \sqrt{x_p^2 + x_q^2}. \)

**Proof.** With \( x = [x_1, x_2, \ldots, x_p, \ldots, x_q, \ldots, x_n]^T \) we have

\[
G_{pq}x = [x_1, x_2, \ldots, x_{p-1}, cx_p + sx_q, x_{p+1}, \ldots, x_{q-1}, -sx_q + cx_p, x_{q+1}, \ldots, x_n]^T
= [a_1, x_2, \ldots, x_{p-1}, \tilde{x}_p, x_{p+1}, \ldots x_{q-1}, 0, x_{q+1}, \ldots x_n]^T = \tilde{x},
\]
as claimed.

**Note.** More generally, we can use the matrix \( G_{pq} \) representing the Givens transformation to transform a symmetric matrix \( X \) into a symmetric matrix \( \tilde{X} \) wit the same rows and columns as \( X \), except for the \( p \)th and \( q \)th rows and columns. We require the \((p,p)\) entry of \( \tilde{X} \) be some given value, say \( \tilde{x}_{pp} = a \). We want a matrix \( G_{pq} \) as given in Theorem 5.2.1 such that \( \tilde{x} = G_{pq}^T X G_{pq} \). We need only consider \( \tilde{x}_{pp} \), since \( G_{pq} \) and \( G_{pq}^T \) will preserve all rows and columns of \( X \), except possible for the \( p \)th and \( q \)th rows and columns. So we need

\[
\begin{bmatrix}
  c & s \\
  -s & c
\end{bmatrix}
\begin{bmatrix}
  x_{pp} & x_{pq} \\
  x_{pq} & x_{qq}
\end{bmatrix}
\begin{bmatrix}
  c & s \\
  -s & c
\end{bmatrix}
= \begin{bmatrix}
  a & \tilde{x}_{pq} \\
  \tilde{x}_{pq} & \tilde{x}_{qq}
\end{bmatrix}
\]

where \( c^2 + s^2 = 1 \). Now

\[
\begin{bmatrix}
  c & -s \\
  s & c
\end{bmatrix}
\begin{bmatrix}
  x_{pp} & x_{pq} \\
  x_{pq} & x_{qq}
\end{bmatrix}
\begin{bmatrix}
  c & s \\
  -s & c
\end{bmatrix}
= \begin{bmatrix}
  cx_{pp} - sx_{pq} & cx_{pq} - sx_{qq} \\
  sx_{pp} + cx_{pq} & sx_{pq} + cx_{qq}
\end{bmatrix}
\begin{bmatrix}
  c & s \\
  -s & c
\end{bmatrix}
\]

\[
\begin{bmatrix}
  c(cx_{pp} - sx_{pq}) - s(cx_{pq} - sx_{qq}) & s(cx_{pp} - sx_{pq}) + c(cx_{pq} - sx_{qq}) \\
  c(sx_{pp} + cx_{pq}) - s(sx_{pq} + cx_{qq}) & s(sx_{pp} + cx_{pq}) + c(sx_{pq} + cx_{qq})
\end{bmatrix}
\]
and so we need

\[ a = c^2 x_{pp} - csq_{pq} - csx_{pq} + s^2 x_{qq} = c^2 x_{pp} - 2csx_{pq} + s^2 x_{qq}. \]

Now

\[
c^2 x_{pp} - 2csx_{pq} + s^2 x_{qq} = c^2(x_{pp} - 2(s/c)x_{pq} + (s/c)^2 x_{qq})
\]

\[= c^2(x_{pp} - 2tx_{pq} + t^2 x_{qq}) \text{ where } t/s/c \]

\[= \frac{1}{1 + t^2}(x_{pp} - 2tx_{pq} + t^2 x_{qq}) \text{ since } 1 + t^2 = 1 + s^2/c^2 \]

\[= (c^2 + s^2)/c^2 \text{ and so } 1/(1 + t^2) = c^2, \]

and so we need \( x_{pp} - 2tx_{pq} + t^2 x_{qq} = (1 + t^2)a \) or

\[(x_{qq} - a)t^2 - 2x_{pq}t + (x_{pp} - a) = 0. \]

The values of \( t \) satisfying this equation are

\[ t = \frac{2x_{pq} \pm \sqrt{4x_{pq}^2 - 4(x_{qq} - a)(x_{pp} - a)}}{2(x_{qq} - a)} = \frac{x_{pq} \pm \sqrt{x_{pq}^2 - (x_{qq} - a)(x_{pp} - a)}}{(x_{qq} - a)}, \]

and there are real solutions if and only if \( x_{pq} \geq (x_{pp} - a)(x_{qq} - a) \). If this condition holds, then choose the nonnegative value of \( t \),

\[ t = \frac{x_{pq} + \sqrt{x_{pq}^2 - (x_{qq} - a)(x_{pp} - a)}}{(x_{qq} - a)}. \]

Next, let \( c = 1/\sqrt{1 + t^2} \) and \( s = ct \). Then \( G_{pq} \) with these values of \( c \) and \( s \) yield the desired transformation. We summarize this in a theorem.
Theorem 5.4.2. Let $X$ be a symmetric matrix and $1 \leq p < q \leq n$. For given $a \in \mathbb{R}$, if $x_{pq}^2 \geq (x_{qq} - a)(x_{pp} - a)$ then with $c = 1/\sqrt{1 + t^2}$ and $s = ct$, where

$$t = \frac{x_{pq} + \sqrt{x_{pq}^2 - (x_{qq} - a)(x_{pp} - a)}}{(x_{qq} - a)},$$

we have $\tilde{X} = G_{pq}^T X G_{pq}$ (where $G_{pq}$ is the matrix from Theorem 5.4.1) is a matrix with the same rows and columns as $X$, except for the $p$th and $q$th rows and columns, and $\tilde{x}_{pp} = a$.

Revised: 5/5/2018