Section 5.6. LU and LDU Factorizations

Note. We largely follow Fraleigh and Beauregard's approach to this topic from *Linear Algebra*, 3rd Edition, Addison-Wesley (1995). See my online notes for Linear Algebra (MATH 2010) on 10.2. The LU-Factorization.

Note. If matrix A can be put in row echelon form without row interchanges (so the only needed elementary row operation is row addition), then there is an upper triangular matrix U and a sequence of elementary $n \times n$ matrices E_i such that $E_h E_{h-1} \cdots E_2 E_1 A = U$ where the diagonal of U is the same as the diagonal of A (see Section 3.2 and Theorem 3.2.3). In addition, each elementary matrix is of the form E_{psq} (representing the row operation $R_p \to R_p + sR_q$) where $p > q$. This is the key observation to showing the existence of an LU-factorization of such a matrix.

Theorem 5.6.A. If A is an $n \times m$ matrix which can be put in row echelon form without interchanging rows then there is a lower triangular $n \times n$ matrix L with entries of 1 on the diagonal and an upper triangular $n \times m$ matrix U such that $A = LU$.

Definition. For $n \times m$ matrix A which can be put in row echelon form without interchanging rows, the factorization $A = LU$ of Theorem 5.6.A is an *LU factorization* of A.

Example 5.6.A. The proof of Theorem 5.6.A gives the algorithm by which the LU-factorization of an appropriate matrix can be found. We row reduce A to row echelon form U and, as each elementary row operation is performed in the reduction of A, perform the inverse of that operation to the identity matrix to produce the corresponding inverse elementary matrix. We have

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 8 & 4 \\ -1 & 3 & 4 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \overset{R_2 \to R_2 - 2R_1}{\longrightarrow} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 6 \\ -1 & 3 & 4 \end{bmatrix} \qquad I \overset{R_2 \to R_2 + 2R_1}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1^{-1}$$

$$\overset{R_3 \to R_3 + R_1}{\longrightarrow} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 6 \\ 0 & 6 & 3 \end{bmatrix} \qquad I \overset{R_3 \to R_3 - R_1}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E_2^{-1}$$

$$\overset{R_3 \to R_3 - 3R_2}{\longrightarrow} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 6 \\ 0 & 0 & -15 \end{bmatrix} \qquad I \overset{R_3 \to R_3 + 3R_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = E_3^{-1}.$$

So we take

$$U = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 6 \\ 0 & 0 & -15 \end{bmatrix} \text{ and } L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}.$$

Notice that in fact $A = LU$.

Note. If $A = LU$ where the diagonal entries of L are all 1, then we can multiply row i of matrix U by $1/u_{ii}$ and produce an upper triangular matrix U^* with diagonal entries of 1. We then create $n \times n$ diagonal matrix D with $d_{ii} = u_{ii}$. This gives $U = DU^*$. We then have a factorization of A as $A = LDU^*$ where the diagonal entries of L and U^* are all 1. The next theorem tells us that when such a factorization of a matrix exists, it is unique.

Theorem 5.6.B. Unique Factorization.
Let A be a square matrix. When a factorization $A = LDU$ exists where

1. L is a lower triangular matrix with all main diagonal entries 1,

2. U is upper triangular matrix with all diagonal entries 1, and

3. D is a diagonal matrix with all main diagonal entries nonzero,

it is unique.

Note. We have from Theorem 5.6.B that, in particular, for square matrix A which can be put in row echelon form without interchanging rows, if $A = LU$ is an LU factorization of A where all diagonal entries of L are 1, then such a factorization is unique. We simply consider $A = L(DU^*)$ where D and U^* are as described in the previous note.

Note. By using a permutation matrix, we can finally address the LU-factorization of a matrix which cannot be put in row echelon form without the use of row interchanges.

Theorem 5.6.C. LU-Factorization.

Let A be and $n \times m$ matrix which can be put in row echelon form. Then there exists a $n \times n$ permutation matrix P, a $n \times n$ lower triangular matrix L, and a $n \times m$ upper triangular matrix U such that $PA = LU$.

Example 7. Consider $\begin{bmatrix} 1 & 3 & 2 \\ -2 & -6 & 1 \\ 2 & 5 & 7 \end{bmatrix}$. We have

$$A \overset{R_2 \to R_2 + 2R_1}{\longrightarrow} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 5 \\ 2 & 5 & 7 \end{bmatrix} \overset{R_3 \to R_3 - 2R_1}{\longrightarrow} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 5 \\ 0 & -1 & 3 \end{bmatrix}.$$

So let $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Then $U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$. The two operations on A which produce U give $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. We then have

$$PA = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 7 \\ -2 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{bmatrix} = LU.$$

Note. Theorem 5.6.A insures that $n \times m$ matrix A which can be put in row echelon form without interchanging rows has an LU factorization. Theorem 5.6.C implies that a square invertible matrix can be modified with a permutation matrix to produce a matrix which has an LU factorization. It is easy to find a square nonsingular matrix which (itself) does not have an LU factorization; consider $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. An example of a nonsquare (and non full rank) matrix with an LU decomposition (given by Gentle on page 188) is

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = LU.$$

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