Section 5.7. QR Factorization

Note. We express a matrix as a product of an orthogonal matrix and an upper triangular matrix. We mostly follow Harville's *Matrix Algebra From a Statistician's Perspective* (Springer, 1997; see pages 66–69). We need a preliminary result from Harville.

Theorem 5.7.A. Let $\{a_1, a_2, \ldots, a_k\}$ be a linearly independent set of vectors. There exists unique scalars x_{ij} where $1 \le j \le k$ and $0 < i < j$ such that the k vectors

$$\begin{aligned} b_1 &= a_1 \\ b_2 &= a_2 - x_{12}b_1 \\ &\vdots \\ b_j &= a_j - x_{j-1,\,j}b_{j-1} - x_{j-2,\,j}b_{j-2} - \cdots - x_{1j}b_1 \\ &\vdots \\ b_m &= a_m - x_{m-1,\,m}b_{m-1} - x_{m-2,\,m}b_{m-2} - \cdots - x_{1m}b_1 \end{aligned}$$

form an orthogonal set. The vectors $b_1, b_2, \ldots, b_k$ are nonzero and $x_{ij} = \langle a_j, b_i\rangle / \langle b_i, b_i\rangle$ for $1 \le j \le m$ and $0 < i < m$.

Note. We present a preliminary type of factorization which we will use in developing QR factorizations.

Theorem 5.7.B. Let A be an $n \times m$ matrix of full column rank (that is, $\text{rank}(A) = m$; so we must have $n \geq m$). Then there is $n \times m$ matrix B, where the columns of B are mutually orthogonal and nonzero, and $m \times m$ upper triangular matrix X, with all diagonal entries 1, such that $A = BX$.

Theorem 5.7.C. Let A be an $n \times m$ matrix of full column rank (that is, $\text{rank}(A) = m$; so we must have $n \geq m$). Then there is a unique factorization of A as $A = BX$ where the columns of $n \times m$ matrix B are mutually orthogonal and nonzero and $m \times m$ matrix X is upper triangular with all diagonal entries 1.

Note. The proof of Theorem 5.7.C is to be given in Exercise 5.7.A.

Note 5.7.A. For full column rank $n \times m$ matrix A with factorization $A = BX$ of Theorem 5.7.C, define $m \times m$ diagonal matrix $D = \text{diag}(\|b_1\|^{-1}, \|b_2\|^{-1}, \ldots, \|b_m\|^{-1})$ and $m \times m$ diagonal matrix $E = \text{diag}(\|b_1\|, \|b_2\|, \ldots, \|b_m\|) = D^{-1}$. With $Q = BD$, the columns of Q are the $b_j/\|b_j\|$ so that Q is an orthogonal matrix. With $R = EX$ we have the (i, j) entry of R is

$$r_{ij} = \begin{cases} \|b_i\| x_{ij} & \text{for } i < j \\ \|b_i\| & \text{for } i = j \\ 0 & \text{for } i > j. \end{cases}$$

Then R is upper triangular with positive entries on the diagonal. Notice $A = BX = BDD^{-1}X = BDEX = QR$.

Definition. For full column rank $n \times m$ matrix A, a factorization $A = QR$ where Q is an $n \times m$ orthogonal matrix and R is an $m \times m$ upper triangular matrix with all diagonal entries positive is a *QR factorization* of A.

Theorem 5.7.D. If A is a $n \times m$ full column rank matrix (that is, $\text{rank}(A) = m$; so we must have $n \geq m$) then there is a unique QR factorization of A.

Note. The proof of Theorem 5.7.D is to be given in Exercise 5.7.B.

Theorem 5.7.E. Let A be a $n \times m$ matrix of rank r where $1 \leq r \leq m$. Then there is a factorization $A = Q_1 R_1$ where Q_1 is an $n \times r$ matrix with orthonormal columns (so Q_1 is orthogonal) and R_1 is an $r \times m$ submatrix of a $m \times m$ upper triangular matrix R having r positive diagonal elements and $m - r$ rows of zeros; the rows of R_1 are the r nonzero rows of R.

Note. Gentle refers to the decomposition of A as $A = Q_1 R_1$ as given in Theorem 5.7.E as a "skinny" QR factorization (see page 189). Gentle addresses "Nonfull Rank Matrices" on pages 189 and 190, but I can't follow it. He is somehow using a permutation matrix to get a result similar to Theorem 5.7.E, though the dimension of the matrices is unclear. I leave it at Theorem 5.7.E to address QR-type factorizations of nonfull rank matrices. The proof of Theorem 5.7.E is left as Exercise 5.7.C.

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