Section 5.7. QR Factorization

Note. We express a matrix as a product of an orthogonal matrix and an upper triangular matrix. We mostly follow Harville’s *Matrix Algebra From a Statistician’s Perspective* (Springer, 1997; see pages 66–69). We need a preliminary result from Harville.

**Theorem 5.7.A.** Let \( \{a_1, a_2, \ldots, a_k\} \) be a linearly independent set of vectors. There exists unique scalars \( x_{ij} \) where \( 1 \leq j \leq k \) and \( 0 < i < j \) such that the \( k \) vectors

\[
\begin{align*}
  b_1 &= a_1 \\
  b_2 &= a_2 - x_{12}b_1 \\
  &\vdots \\
  b_j &= a_j - x_{j-1,j}b_{j-1} - x_{j-2,j}b_{j-2} - \cdots - x_{1j}b_1 \\
  &\vdots \\
  b_m &= a_m - x_{m-1,m}b_{m-1} - x_{m-2,m}b_{m-2} - \cdots - x_{1m}b_1
\end{align*}
\]

form an orthogonal set. The vectors \( b_1, b_2, \ldots, b_k \) are nonzero and \( x_{ij} = \langle a_j, b_i \rangle / \langle b_i, b_i \rangle \) for \( 1 \leq j \leq m \) and \( 0 < i < m \).

Note. We present a preliminary type of factorization which we will use in developing QR factorizations.
Theorem 5.7.B. Let $A$ be an $n \times m$ matrix of full column rank (that is, $\text{rank}(A) = m$; so we must have $n \geq m$). Then there is $n \times m$ matrix $B$, where the columns of $B$ are mutually orthogonal and nonzero, and $m \times m$ upper triangular matrix $X$, with all diagonal entries 1, such that $A = BX$.

Theorem 5.7.C. Let $A$ be an $n \times m$ matrix of full column rank (that is, $\text{rank}(A) = m$; we must have $n \geq m$). Then there is a unique factorization of $A$ as $A = BX$ where the columns of $n \times m$ matrix $B$ are mutually orthogonal and nonzero and $m \times m$ matrix $X$ is upper triangular with all diagonal entries 1.

Note. The proof of Theorem 5.7.C is to be given in Exercise 5.7.A.

Note 5.7.A. For full column rank $n \times m$ matrix $A$ with factorization $A = BX$ of Theorem 5.7.C, define $m \times m$ diagonal matrix $D = \text{diag}(\|b_1\|^{-1}, \|b_2\|^{-1}, \ldots, \|b_m\|^{-1})$ and $m \times m$ diagonal matrix $E = \text{diag}(\|b_1\|, \|b_2\|, \ldots, \|b_m\|) = D^{-1}$. With $Q = BD$, the columns of $Q$ are the $b_j/\|b_j\|$ so that $Q$ is an orthogonal matrix. With $R = EX$ we have the $(i, j)$ entry of $R$ is

$$r_{ij} = \begin{cases} \|b_i\| x_{ij} & \text{for } j > i \\ \|b_i\| & \text{for } j = i \\ 0 & \text{for } j < i. \end{cases}$$

Then $R$ is upper triangular with positive entries on the diagonal. Notice $A = BX = BDD^{-1}X = BDEX = QR$. 
**Definition.** For full column rank $n \times m$ matrix $A$, a factorization $A = QR$ where $Q$ is an $n \times m$ orthogonal matrix and $R$ is an $m \times m$ upper triangular matrix with all diagonal entries positive is a $QR$ factorization of $A$.

**Theorem 5.7.D.** If $A$ is a $n \times m$ full column rank matrix (that is, $\text{rank}(A) = m$; so we must have $n \geq m$) then there is a unique $QR$ factorization of $A$.

**Note.** The proof of Theorem 5.7.D is to be given in Exercise 5.7.B.

**Theorem 5.7.E.** Let $A$ be a $n \times m$ matrix of rank $r$ where $1 \leq r \leq m$. Then there is a factorization $A = Q_1R_1$ where $Q_1$ is an $n \times r$ matrix with orthonormal columns (so $Q_1$ is orthogonal) and $R_1$ is an $r \times m$ submatrix of a $m \times m$ upper triangular matrix $R$ having $r$ positive diagonal elements and $m - r$ rows of zeros; the rows of $R_1$ are the $r$ nonzero rows of $R$.

**Note.** Gentle refers to the decomposition of $A$ as $A = Q_1R_1$ as given in Theorem 5.7.E as a “skinny” $QR$ factorization (see page 189). Gentle addresses “Nonfull Rank Matrices” on pages 189 and 190, but I can’t follow it. He is somehow using a permutation matrix to get a result similar to Theorem 5.7.E, though the dimension of the matrices is unclear. I leave it at Theorem 5.7.E to address $QR$-type factorizations of nonfull rank matrices. The proof of Theorem 5.7.E is left as Exercise 5.7.C.

*Revised: 5/12/2018*