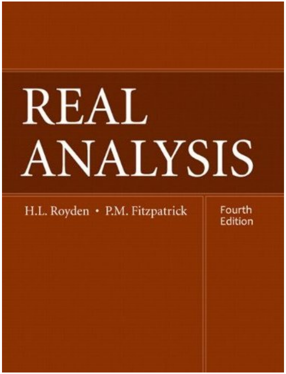
Real Analysis

Chapter 10. Metric Spaces: Three Fundamental Theorem

10.1. The Arzelà-Ascoli Theorem—Proofs of Theorems



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Proof. Let $\{x_j\}_{j=1}^{\infty}$ be an enumeration of a dense subset D of separable metric space X. Since $\{f_n\}$ is pointwise bounded, the the sequence of real numbers defined by $n \mapsto f_n(x_1)$ is bounded. Therefore, by the Bolzano-Weierstrass Theorem (see my online notes for Analysis 1 [MATH 4217/5217] on Section 2.3. Bolzano-Weierstrass Theorem) this sequence has a convergent subsequence. That is, there is a strictly increasing sequence of natural numbers $\{s(1,n)\}$ (which act as indices) and a umber a_1 for which $\lim_{n\to\infty} f_{s(1,n)}(x_1) = a_1$. Similarly, the sequence $n \mapsto f_{s(1,n)}(x_2)$ is bounded and so there is a subsequence $\{s(2,n)\}$ of $\{s(1,n)\}$ and a number a_2 for which $\lim_{n\to\infty} f_{s(2,n)}(x_2) = a_2$.

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Proof (continued). Continuing inductively, we obtain a countable collection of strictly increasing sequences of natural numbers $\{\{s(j,n)\}\}_{j=1}^{\infty}$ and a sequence of numbers $\{a_j\}$ such that for each j,

$$\{s(j+1,n)\} \text{ is a subsequence of } \{s(j,n)\} \text{ and } \lim_{n\to\infty} f_{s(j,n)}(x_j) = a_j.$$

Next, we define function f. First, for each index j, define $f(x_j) = a_j$. Consider the "diagonal" subsequence (of sequence $\{f_n\}$) $\{f_{n_k}\}$ obtained by setting $n_k = s(k,k)$ for each index k. For each j, $\{n_k\}_{k=j}^{\infty} = \{n_{s(k,k)}\}_{k=j}^{\infty}$ is a subsequence of the jth subsequence of natural numbers $\{s(j,n)\}$. Since $\lim_{n\to\infty} f_{s(j,n)}(x_j) = a_j$, then $\lim_{k\to\infty} f_{x_k}(x_j) = a_j = f(x_j)$. Thus $\{f_{n_k}\}$ converges pointwise on D to f.

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Let X be a separable metric space and $\{f_n\}$ an equicontinuous sequence in $C(X)$ that is pointwise bounded. Then a subsequence of $\{f_n\}$ converges pointwise on all of X to a real-valued function f on X.

Proof (continued). For "notational convenience," assume the whole sequence of $\{f_n\}$ converges pointwise on F to f (so that we don't have to refer to subsequence $\{f_{n_k}\} = \{f_{n_{s(k,k)}}\}$). Let x_0 be any point in X. We claim that $\{f_n(x_0)\}$ is Cauchy. Let $\varepsilon > 0$. By the equicontinuity of $\{f_n\}$ at x_0, we may choose $\delta > 0$ such that $|f_n(x) - f_n(x_0)| < \varepsilon/3$ for all indices n and all $x \in X$ for which $\rho(x, x_0) < \delta$. Since D is dense, there is a point $x \in D$ such that $\rho(x, x_0) < \delta$. Moreover, since sequence $\{f_n(x)\}$ converges (to $f(x)$ since $x \in D$), then it must be Cauchy, so that we can choose N so large that

$$|f_n(x) - f_m(x)| < \varepsilon \text{ for all } m, n \geq N.$$

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Proof (continued). Then for all $m, n \geq N$ we have by the Triangle Inequality on $\mathbb{R}$ that

$$\begin{aligned} |f_n(x_0) - f_m(x_0)| &\leq |f_n(x_0) - f_n(x)| + |f_n(x) - f_m(x)| + |f_m(x) - f_m(x_0)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Thus sequence $\{f_n(x_0)\}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, the sequence converges. Denote the limit by $f(x_0)$. The sequence $\{f_n\}$ converges pointwise on all of X to $f : X \to \mathbb{R}$. (Notice that we have not claimed that f is in $C(X)$, so we are done.) □

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Let X be a compact metric space and $\{f_n\}$ is a uniformly bounded, equicontinuous sequence of real-valued functions on X. Then $\{f_n\}$ has a subsequence that converges uniformly on X to a continuous function f on X.

Proof. Since X is a compact metric space, then by Proposition 9.24 X is separable. So we can apply the Arzelà-Ascoli Lemma (Lemma 9.2) to $\{f_n\}$ (since the sequence is equicontinuous, then each f_n is in $C(X)$) to conclude that it has a subsequence that converges pointwise on all of X to a real-valued function f. Again for notational convenience, assume the whole sequence $\{f_n\}$ converges pointwise on X (to avoid repeated reference to the subsequence). Therefore, for each $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence of real numbers. We use this and equicontinuity to show that $\{f_n\}$ is a Cauchy sequence in $C(X)$.

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Theorem 10.1.A. The Arzelà-Ascoli Theorem (continued 1)

Proof (continued). Let $\varepsilon > 0$. By the uniform equicontinuity of $\{f_n\}$ on X, there is a $\delta > 0$ such that for all $n \in \mathbb{N}$

$$|f_n(u) - f_n(v)| < \varepsilon/3 \text{ for all } u, v \in X \text{ such that } \rho(u, v) < \delta. \qquad (1)$$

Since X is a compact metric space, by the Characterization of Compactness for a Metric Space (Theorem 9.16; the (ii) implies (i) part)), then X is totally bounded. Therefore there is a finite number of points $x_1, x_2, \ldots, x_k$ in X for which X is covered by $\{B(x_i, \delta)\}_{i=1}^k$. For $1 \le i \le k$, $\{f_n(x_i)\}$ is a Cauchy sequence of real numbers (since it converges to $f(x)$), so there is an index $N \in \mathbb{N}$ such that

$$|f_n(x_i) - f_m(v)| < \varepsilon/3 \text{ for } 1 \le i \le k \text{ and all } n, m \ge N. \qquad (2)$$

Since $\{B(x_i, \delta)\}_{i=1}^k$ covers X, then for any $x \in X$ there is an i with $1 \le i \le k$, such that $\rho(x, x_i) < \delta$.

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Proof (continued). Therefore for $n, m \geq N$, we have by the Triangle Inequality, (1), and (2) that

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Thus $\{f_n\}$ is uniformly Cauchy. Therefore, since $C(X)$ is complete, then $\{f_n\}$ converges uniformly on X to some function f. A uniform limit of a sequence of continuous functions is continuous, so that the limit function f is continuous on X, as claimed. □

Theorem 10.3

Theorem 10.3. Let X be a compact metric space and $\mathcal{F}$ a subset of $C(X)$. Then $\mathcal{F}$ is a compact subspace of $C(X)$ if and only if $\mathcal{F}$ is closed, uniformly bounded, and equicontinuous.

Proof. First, suppose that $\mathcal{F}$ is closed, uniformly bounded, and equicontinuous. Let $\{f_n\}$ be a sequence in $\mathcal{F}$. By the Arzelà-Ascoli Theorem (Theorem 10.1.A), a subsequence of $\{f_n\}$ converges uniformly to a function $f \in C(X)$. Since $\mathcal{F}$ is closed, then $f \in \mathcal{F}$. Since $\{f_n\}$ is an arbitrary sequence in $\mathcal{F}$, then (by definition of "sequentially compact") $\mathcal{F}$ is sequentially compact and, therefore, compact by Characterization of Compactness for a Metric Space (Theorem 9.16; the (iii) implies (ii) part), as claimed.

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Second, suppose that $\mathcal{F}$ is compact. We leave it as an exercise (Problem 10.1.A) to show that $\mathcal{F}$ is uniformly bounded and is a closed subset of $C(X)$, so we now only need to show equicontinuity of $\mathcal{F}$. ASSUME that $\mathcal{F}$ is not equicontinuous at some point $x \in X$.

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Theorem 10.3 (continued 1)

Proof (continued). Then there is an $\varepsilon_0 > 0$ such that for each $n \in \mathbb{N}$, there is a function in $\mathcal{F}$, which we denote as f_n, and a point $x_n \in X$ for which

$$|f_n(x_n) - f_n(x)| \geq \varepsilon_0 \text{ even though } \rho(x_n, x) < 1/n. \qquad (3)$$

This gives us a sequence $\{f_n\}$ in $\mathcal{F}$. Since $\mathcal{F}$ is a compact metric space, then it is sequentially compact by the Characterization of Compactness for a Metric Space (Theorem 9.16; the (ii) implies (iii) part). Therefore there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges on X. Since $\mathcal{F} \subseteq C(X)$ and $C(X)$ has norm $\rho_{\max}$, then the convergence of sequence $\{f_{n_k}\}$ to $\{f_n\}$ is uniform. Therefore the limit of $\{f_{n_k}\}$ is a function f continuous on X; that is, $f \in C(X)$.

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Proof (continued). Since $\{f_{n_k}\}$ converges to f in $C(X)$ under $\rho_{\max}$, then we can choose an index K such that $\rho_{\max}(f, f_{n_k}) < \varepsilon_0/3$ for $k \geq K$. By replacing x_n with x_{n_k} in (3), we have

$$|f_n(x_{n_k}) - f(x)| \geq \varepsilon_0/3 \text{ even though } \rho(x_{n_k}, x) < 1/n_k. \qquad (4)$$

But this CONTRADICTS the fact that f is continuous at point $x \in X$. This contradiction shows that the assumption that $\mathcal{F}$ is not equicontinuous at $x \in X$ is false. Therefore $\mathcal{F}$ is equicontinuous on X, as claimed. □