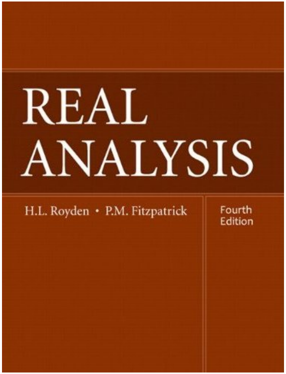
Real Analysis

Chapter 10. Metric Spaces: Three Fundamental Theorem

10.2. The Baire Category Theorem—Proofs of Theorems



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(i) Let $\{\mathcal{O}_n\}_{n-1}^{\infty}$ be a countable collection of open dense subsets of X. Then the intersection $\cap_{n=1}^{\infty}\mathcal{O}_n$ is also dense.

(ii) Let $\{F_n\}_{n=1}^{\infty}$ be a countable collection of closed hollow subsets of X. Then the union $\cup_{n=1}^{\infty}F_n$ also is hollow.

Proof. A set is dense if and only if its complement is hollow by Note 10.2.A. A set is open if and only if its complement is closed. Then by De Morgan's Identities, (i) and (ii) are equivalent, so it is sufficient to prove (i). Let $x_0 \in X$ and let $r_0 > 0$. To show that $\cap_{n=1}^{\infty}\mathcal{O}_n$ is dense, we need to show that $B(x_0, r_0)$ contains a point of $\cap_{n=1}^{\infty}\mathcal{O}_n$.

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Since $\mathcal{O}_1$ is dense in X, then there is some $x_1 \in B(x_0, r_0) \cap \mathcal{O}_1$. Choose r_1 where $0 < r_1 < 1$ for which $\overline{B}_1(x_1, r_1) \subseteq B(x_0, r_0) \cap \mathcal{O}_1$ (this can be done by Note 10.2.B); denote $B_1 = B(x_1, r_1)$.

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Theorem 10.2.A. The Baire Category Theorem (continued)

Proof (continued). We now inductively create a contracting sequence of closed balls, the intersection of which will give and element of $\cap_{n=1}^{\infty}\mathcal{O}_n$ in $B(x_0, r_0)$. Suppose $n \in \mathbb{N}$ and the descending collection of open balls $\{B_k\}_{k=1}^{n}$ has been chosen with the property that for $1 \leq k \leq n$, B_k has radius less than $1/k$ and $\overline{B}_k \subseteq \mathcal{O}_k$ (the base case if given by B_1 above). The set $B_n \cap \mathcal{O}_{n+1}$ is nonempty since $\mathcal{O}_{n+1}$ is dense in X. Let x_{n+1} belong to the open set $B(x_n, r_n) \cap \mathcal{O}_n$. Choose r_{n+1} such that $0 < r_{n+1} < 1/(n+1)$ for which $\overline{B}_{n+1}(x_{n+1}, r_{n+1}) \subseteq B_n \cap \mathcal{O}_{n+1}$ (this can be done by Note 10.2.B); denote $B_{n+1} = B(x_{n+1}, r_{n+1})$. This inductively defines a contracting sequence of closed sets $\{\overline{B}_n\}_{n=1}^{\infty}$ with the property that for each n, $\overline{B}_n \subseteq \mathcal{O}_n$. The metric space X is complete by hypothesis, so from the Cantor Intersection Theorem we have that $\cap_{n=1}^{\infty}\overline{B}_n$ is nonempty.

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Proof (continued). We now inductively create a contracting sequence of closed balls, the intersection of which will give and element of $\cap_{n=1}^{\infty}\mathcal{O}_n$ in $B(x_0, r_0)$. Suppose $n \in \mathbb{N}$ and the descending collection of open balls $\{B_k\}_{k=1}^{n}$ has been chosen with the property that for $1 \leq k \leq n$, B_k has radius less than $1/k$ and $\overline{B}_k \subseteq \mathcal{O}_k$ (the base case if given by B_1 above). The set $B_n \cap \mathcal{O}_{n+1}$ is nonempty since $\mathcal{O}_{n+1}$ is dense in X. Let x_{n+1} belong to the open set $B(x_n, r_n) \cap \mathcal{O}_n$. Choose r_{n+1} such that $0 < r_{n+1} < 1/(n+1)$ for which $\overline{B}_{n+1}(x_{n+1}, r_{n+1}) \subseteq B_n \cap \mathcal{O}_{n+1}$ (this can be done by Note 10.2.B); denote $B_{n+1} = B(x_{n+1}, r_{n+1})$. This inductively defines a contracting sequence of closed sets $\{\overline{B}_n\}_{n=1}^{\infty}$ with the property that for each n, $\overline{B}_n \subseteq \mathcal{O}_n$. The metric space X is complete by hypothesis, so from the Cantor Intersection Theorem we have that $\cap_{n=1}^{\infty}\overline{B}_n$ is nonempty. Let x_* belong to this intersection. Then $x_* \in \cap_{n=1}^{\infty}\mathcal{O}_n$. Since $\overline{B}_1 \subseteq B(x_0, r_0) \cap \mathcal{O}_1$, then we have $x_* \in B(x_0, r_0)$ also. Open ball $B(x_0, r_0)$ is an arbitrary open ball and we have that $b(x_0, r_0)$ contains a point of $\cap_{n=1}^{\infty}\mathcal{O}_n$, so that $\cap_{n=1}^{\infty}\mathcal{O}_n$ is dense, as claimed. □

Corollary 10.5

Corollary 10.5. Let X be a complete metric space and $\{F_n\}_{n=1}^{\infty}$ a countable collection of closed subsets of X. Then $\cup_{n=1}^{\infty} \mathrm{bd}(F_n)$ is hollow.

Proof. In Problem 10.15 it is to be shown that the boundary of any set is itself a closed set, and the boundary of a closed set E is hollow. Therefore, for each $n \in \mathbb{N}$, $\mathrm{bd}(F_n)$ is closed and hollow. Then by the Baire Category Theorem (Theorem 10.2.A), $\cup_{n=1}^{\infty} \mathrm{bd}(F_n)$ is hollow, as claimed. □

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Theorem 10.6

Theorem 10.6. Let $\mathcal{F}$ be a family of continuous real-valued functions on a complete metric space X that is pointwise bounded in the sense that for each $x \in X$, there is a constant M_x for which $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$. Then there is a nonempty open subset $\mathcal{O}$ of X on which $\mathcal{F}$ is uniformly bounded in the sense that there is a constant M for which $|f| \leq M$ on $\mathcal{O}$ for all $f \in \mathcal{F}$.

Proof. For each $n \in \mathbb{N}$, define $E_n = \{x \in X \mid |f(x)| \leq n \text{ for all } f \in \mathcal{F}\}$. Then $f^{-1}([-n, n])$ is closed for each function in $\mathcal{F}$ since each such function is continuous (by Problem 9.36) and hence $E_n = \cap_{f \in \mathcal{F}} f^{-1}(-n, n])$ is closed. Since $\mathcal{F}$ is pointwise bounded by hypothesis, then for each $x \in X$ there is $n \in \mathbb{N}$ such that $|f(x)| \leq n$ for all $f \in \mathcal{F}$; that is, $x \in E_n$. Hence, since x is an arbitrary element of X, we have $X = \cup_{n=1}^{\infty} E_n$.

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Theorem 10.7

Theorem 10.7. Let X be a complete metric space and $\{f_n\}$ a sequence of continuous real-valued functions on X that converges pointwise on X to the real-valued function f. Then there is a dense subset D of X for which $\{f_n\}$ is equicontinuous at each point in D.

Proof. Let $m, n \in \mathbb{N}$. Define

$$E(m,n) = \{x \in X \mid |f_j(x) - f_k(x)| \leq 1/m \text{ for all } j,k \geq n\}.$$

The mapping $x \mapsto |f_j(x) - f_k(x)|$ is continuous, and $\{x \in X \mid |f_j(x) - f_k(x)| \leq 1/m\}$ is the inverse image of $[-1/m, 1/m]$ under this mapping so that this set is closed (by Problem 9.36). Now $E(m,n) = \cap_{j,k \geq n}\{x \in X \mid |f_j(x) - f_k(x)| \leq 1/m\}$ so $E(m,n)$ is closed. Then by Corollary 10.5, $\cup_{n,m \in \mathbb{N}} \text{bd}(E(m,n))$ is hollow and, by Note 10.2.A, $D = X \sim [\cup_{n,m \in \mathbb{N}} \text{bd}(E(m,n))]$ is dense in X. If $n, m \in \mathbb{N}$ and point $x \in D$ belongs to $E(m,n)$, then x belongs to the interior of $E(m,n)$ because D contains all boundary points of $E(m,n)$.

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Theorem 10.7 (continued 1)

Proof (continued). We now show that $\{f_n\}$ is equicontinuous at each point of D. Let $x_0 \in D$ and let $\varepsilon > 0$. Choose $m \in \mathbb{N}$ for which $1/m < \varepsilon/4$. Since $\{f_n(x_0)\}$ converges to to real number $f(x_0)$, then $\{f_n(x_0)\}$ is Cauchy. Hence we can choose a natural number N for which

$$|f_j(x_0) - f_k(x_0)| \leq 1/m \text{ for all } j, k \geq N. \qquad (9)$$

Therefore $x_0 \in E(m, N)$. Since, as observed above, x_0 belongs to the interior of $E(m, N)$. Choose $r > 0$ such that $B(x_0, r) \subseteq E(m, N)$; that is,

$$|f_j(x) - f_k(x)| \leq 1/m \text{ for all } j, k \geq N \text{ and for all } x \in B(x_0, r). \qquad (10)$$

The function f_N is continuous at x_0. Therefore there is a $\delta > 0$ with $0 < \delta < r$ for which

$$|f_N(x) - f_N(x)| < 1/m \text{ for all } x \in B(x_0, \delta). \qquad (11)$$

Theorem 10.7 (continued 2)

Proof (continued). Notice that for all $x \in X$ and $j \in \mathbb{N}$ we have

$$f_j(x) - f_j(x_0) = [f_j(x) - f_N(x)] + [f_N(x) - f_N(x_0)] + [f_N(x_0) - f_j(x_0)].$$

We now have by the Triangle Inequality, (9), (10), (11), and the fact that $1/m < \varepsilon/4$ that

$$|f_j(x) - f_j(x_0)| \leq 3/m < (3/4)\varepsilon \text{ for all } j \geq N \text{ and for all } x \in B(x_0, \delta). \quad (12)$$

The finite family of continuous functions $\{f_j\}_{j=1}^{N-1}$ is equicontinuous at x_0 (we just choose the smallest of the associated δ's). Combining this fact with (12), we have that $\{f_n\}$ is equicontinuous at x_0. Since x_0 is an arbitrary point of D, then $\{f_n\}$ is equicontinuous on D, as claimed. Since $\{f_n\}$ converges pointwise to f by hypothesis, then by taking the limit as $j \to \infty$ in (12), we have $|f(x) - f(x_0)| < \varepsilon$ for all $x \in B(x_0, \delta)$. That is, f is continuous at x_0. Since x_0 is an arbitrary point of D, then f is continuous on dense subset D, as claimed. □

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