Real Analysis

Chapter 10. Metric Spaces: Three Fundamental Theorem

10.3. The Banach Contraction Principle—Proofs of Theorems

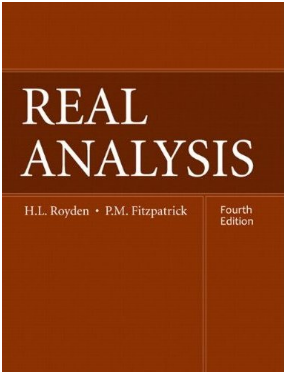

Table of contents

Theorem 10.3.B. The Banach Contraction Principle

Theorem 10.3.B. The Banach Contraction Principle.
Let X be a complete metric space and the mapping $T : X \to X$ be a contract. Then $T : X \to X$ has exactly one fixed point.

Proof. Let c be a number with $0 \leq c < 1$ that is a Lipschitz constant for mapping T. Select any point in X and label it x_0. Define the sequence $\{x_k\}$ inductively be defining $x_1 = T(0_x)$ and, if $k \in \mathbb{N}$ is such that x_k is defined, then define $x_{k+1} = T(x_k)$. The sequence $\{x_n\}$ is properly defined since $T(x)$ is a subset of X. We will show that $\{x_n\}$ converges to a fixed point of T.

Theorem 10.3.B. The Banach Contraction Principle

Theorem 10.3.B. The Banach Contraction Principle.
Let X be a complete metric space and the mapping $T : X \to X$ be a contract. Then $T : X \to X$ has exactly one fixed point.

Proof. Let c be a number with $0 \leq c < 1$ that is a Lipschitz constant for mapping T. Select any point in X and label it x_0. Define the sequence $\{x_k\}$ inductively be defining $x_1 = T(0_x)$ and, if $k \in \mathbb{N}$ is such that x_k is defined, then define $x_{k+1} = T(x_k)$. The sequence $\{x_n\}$ is properly defined since $T(x)$ is a subset of X. We will show that $\{x_n\}$ converges to a fixed point of T.

Since T is a contraction with Lipschitz constant c, we have

$$\rho(x_2, x_1) = \rho(T(x_1), T(x_0)) = \rho(T(T(x_0)), T(x_0)) \leq c\,\rho(T(x_0), x_0),$$

and similarly

$$\rho(x_{k+1}, x_k) = \rho(T(x_k), T(x_{k-1})) \leq x\,\rho(x_k, x_{k-1}) \text{ if } k \geq 2.$$

Theorem 10.3.B. The Banach Contraction Principle

Theorem 10.3.B. The Banach Contraction Principle.
Let X be a complete metric space and the mapping $T : X \to X$ be a contract. Then $T : X \to X$ has exactly one fixed point.

Proof. Let c be a number with $0 \leq c < 1$ that is a Lipschitz constant for mapping T. Select any point in X and label it x_0. Define the sequence $\{x_k\}$ inductively be defining $x_1 = T(0_x)$ and, if $k \in \mathbb{N}$ is such that x_k is defined, then define $x_{k+1} = T(x_k)$. The sequence $\{x_n\}$ is properly defined since $T(x)$ is a subset of X. We will show that $\{x_n\}$ converges to a fixed point of T.

Since T is a contraction with Lipschitz constant c, we have

$$\rho(x_2, x_1) = \rho(T(x_1), T(x_0)) = \rho(T(T(x_0)), T(x_0)) \leq c\,\rho(T(x_0), x_0),$$

and similarly

$$\rho(x_{k+1}, x_k) = \rho(T(x_k), T(x_{k-1})) \leq x\,\rho(x_k, x_{k-1}) \text{ if } k \geq 2.$$

Theorem 10.3.B (continued 1)

Proof (continued). These two inequalities imply, by induction, that

$$\rho(x_{k+1}, x_k) \leq c^k \rho(T(x_0), x_0) \text{ for every } k \in \mathbb{N}.$$

Hence if $m, k \in \mathbb{N}$ with $m > k$, then

$$\begin{aligned}
\rho(x_m, x_k) &\leq \rho(x_m, x_{m-1}) + \rho(x_{m-1}, x_{m-2}) + \cdots + \rho(x_{k+1}, x_k) \\
&\qquad \text{by the Triangle Inequality for } \rho \\
&\leq (c^{m-1} + c^{m-2} + \cdots + c^k)\rho(T(x_0), x_0) \\
&= c^k(1 + c + \cdots + c^{m-1-k})\rho(T(x_0), x_0) \\
&= c^k \frac{1 - c^{m-k}}{1 - c}\rho(T(x_0), x_0) \text{ since } \sum_{k=0}^{n} c^k = \frac{1 - c^{n+1}}{1 - c} \text{ for } c \neq 1.
\end{aligned}$$

Since $0 \leq c < 1$, then $\rho(x_m, x_k) \leq \dfrac{c^k}{1 - c}\rho(T(x_0), x_0)$ if $m > k$.

Theorem 10.3.B (continued 2)

Proof (continued). Since $0 \le c < 1$, then $\rho(x_m, x_k) \le \dfrac{c^k}{1-c}\rho(T(x_0), x_0)$ if $m > k$. But $\lim_{k\to\infty} c^k = 0$, so that the inequality implies that $\{x_k\}$ is a Cauchy sequence.

By hypothesis, the metric space S is complete. Thus there is a point $x \in X$ to which the sequence $\{x_k\}$ converges. Since T is Lipschitz, it is continuous (let $\delta = \varepsilon/c$ in the ϵ/δ Criterion for Continuity, Theorem 9.3.A). Therefore $T(x) = \lim_{k\to\infty} T(x_k) = T(\lim_{k\to\infty} x_k) = \lim_{k\to\infty} x_{k+1} = x$. Thus the mapping $T : X \to X$ has at least one fixed point.

Theorem 10.3.B (continued 2)

Proof (continued). Since $0 \leq c < 1$, then $\rho(x_m, x_k) \leq \dfrac{c^k}{1-c}\rho(T(x_0), x_0)$ if $m > k$. But $\lim_{k\to\infty} c^k = 0$, so that the inequality implies that $\{x_k\}$ is a Cauchy sequence.

By hypothesis, the metric space S is complete. Thus there is a point $x \in X$ to which the sequence $\{x_k\}$ converges. Since T is Lipschitz, it is continuous (let $\delta = \varepsilon/c$ in the ϵ/δ Criterion for Continuity, Theorem 9.3.A). Therefore
$T(x) = \lim_{k\to\infty} T(x_k) = T\left(\lim_{k\to\infty} x_k\right) = \lim_{k\to\infty} x_{k+1} = x$. Thus the mapping $T : X \to X$ has at least one fixed point. ASSUME that u and v are two different fixed points such that $T(u) = u$ and $T(v) = v$. Then $0 \leq \rho(u, v) = \rho(T(u), T(v)) \leq c\,\rho(u, v)$, so that since $0 \leq c < 1$, we must have $\rho(u, v) = 0$. But then $u = v$, a CONTRADICTION. So the assumption that there are two different fixed points is false, and hence T has exactly one fixed point, as claimed. □

Theorem 10.3.B (continued 2)

Proof (continued). Since $0 \le c < 1$, then $\rho(x_m, x_k) \le \dfrac{c^k}{1-c}\rho(T(x_0), x_0)$ if $m > k$. But $\lim_{k\to\infty} c^k = 0$, so that the inequality implies that $\{x_k\}$ is a Cauchy sequence.

By hypothesis, the metric space S is complete. Thus there is a point $x \in X$ to which the sequence $\{x_k\}$ converges. Since T is Lipschitz, it is continuous (let $\delta = \varepsilon/c$ in the ϵ/δ Criterion for Continuity, Theorem 9.3.A). Therefore
$T(x) = \lim_{k\to\infty} T(x_k) = T\left(\lim_{k\to\infty} x_k\right) = \lim_{k\to\infty} x_{k+1} = x$. Thus the mapping $T : X \to X$ has at least one fixed point. ASSUME that u and v are two different fixed points such that $T(u) = u$ and $T(v) = v$. Then $0 \le \rho(u, v) = \rho(T(u), T(v)) \le c\,\rho(u, v)$, so that since $0 \le c < 1$, we must have $\rho(u, v) = 0$. But then $u = v$, a CONTRADICTION. So the assumption that there are two different fixed points is false, and hence T has exactly one fixed point, as claimed. □

Theorem 10.3.C. The Picard Local Existence Theorem

Theorem 10.3.C. The Picard Local Existence Theorem.
Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ containing the point (x_0, y_0). Suppose the function $g : \mathcal{O} \to \mathbb{R}^2$ is continuous and there is a positive number M for which the following Lipschitz property in the second variable holds uniformly with respect to the first variable:

$$|g(x, y_1) - g(x, y_2)| \leq M|y_1 - y_2| \text{ for all points } (x, y_1) \text{ and } (x, y_2) \text{ in } \mathcal{O}. \quad (16)$$

Then there is an open interval I containing x_0 on which the following differential equation has a unique solution:

$$\begin{aligned} f'(x) &= g(x, f(x)) \text{ for all } x \in I \\ f(x_0) &= y_0. \end{aligned} \quad (14)$$

Proof. For ℓ a positive number, define I_ℓ to be the closed interval $[x_0 - \ell, x_0 + \ell]$. By Note 10.3.C, we have a solution of differential equation (14) if and only if we have a solution of integral equation (15).

Theorem 10.3.C. The Picard Local Existence Theorem

Theorem 10.3.C. The Picard Local Existence Theorem.
Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ containing the point (x_0, y_0). Suppose the function $g : \mathcal{O} \to \mathbb{R}^2$ is continuous and there is a positive number M for which the following Lipschitz property in the second variable holds uniformly with respect to the first variable:

$$|g(x, y_1) - g(x, y_2)| \leq M|y_1 - y_2| \text{ for all points } (x, y_1) \text{ and } (x, y_2) \text{ in } \mathcal{O}. \quad (16)$$

Then there is an open interval I containing x_0 on which the following differential equation has a unique solution:

$$\begin{aligned} f'(x) &= g(x, f(x)) \text{ for all } x \in I \\ f(x_0) &= y_0. \end{aligned} \quad (14)$$

Proof. For ℓ a positive number, define I_ℓ to be the closed interval $[x_0 - \ell, x_0 + \ell]$. By Note 10.3.C, we have a solution of differential equation (14) if and only if we have a solution of integral equation (15).

Theorem 10.3.C (continued 1)

Proof (continued). So, it suffices to show that ℓ can be chosen so that there is exactly one continuous function $f : I_\ell \to \mathbb{R}$ with the property that

$$f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt \text{ for all } x \in I_\ell.$$

Since $\mathcal{O}$ is open, we may choose positive numbers a and b such that the closed rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ is contained in $\mathcal{O}$. Now for each positive number ℓ with $\ell \leq a$, define X_ℓ to be the subspace of the metric space $C(I_\ell)$ consisting of those continuous functions $f : I_\ell \to \mathbb{R}$ such that $|f(x) - y_0| \leq b$ for all $x \in I_\ell$. That is, X_ℓ consists of all continuous functions on I_ℓ (since X_ℓ is a subspace of $C(I_\ell)$) that have a graph contained in the rectangle $I_\ell \times [y_0 - b, y_0 + b]$.

Theorem 10.3.C (continued 1)

Proof (continued). So, it suffices to show that ℓ can be chosen so that there is exactly one continuous function $f : I_\ell \to \mathbb{R}$ with the property that

$$f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt \text{ for all } x \in I_\ell.$$

Since $\mathcal{O}$ is open, we may choose positive numbers a and b such that the closed rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ is contained in $\mathcal{O}$. Now for each positive number ℓ with $\ell \leq a$, define X_ℓ to be the subspace of the metric space $C(I_\ell)$ consisting of those continuous functions $f : I_\ell \to \mathbb{R}$ such that $|f(x) - y_0| \leq b$ for all $x \in I_\ell$. That is, X_ℓ consists of all continuous functions on I_ℓ (since X_ℓ is a subspace of $C(I_\ell)$) that have a graph contained in the rectangle $I_\ell \times [y_0 - b, y_0 + b]$.

For $f \in X_\ell$, define the function $T(f) \in C(I_\ell)$ by

$$T(f)(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt \text{ for all } x \in I_\ell.$$

Theorem 10.3.C (continued 1)

Proof (continued). So, it suffices to show that ℓ can be chosen so that there is exactly one continuous function $f : I_\ell \to \mathbb{R}$ with the property that

$$f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt \text{ for all } x \in I_\ell.$$

Since $\mathcal{O}$ is open, we may choose positive numbers a and b such that the closed rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ is contained in $\mathcal{O}$. Now for each positive number ℓ with $\ell \leq a$, define X_ℓ to be the subspace of the metric space $C(I_\ell)$ consisting of those continuous functions $f : I_\ell \to \mathbb{R}$ such that $|f(x) - y_0| \leq b$ for all $x \in I_\ell$. That is, X_ℓ consists of all continuous functions on I_ℓ (since X_ℓ is a subspace of $C(I_\ell)$) that have a graph contained in the rectangle $I_\ell \times [y_0 - b, y_0 + b]$.

For $f \in X_\ell$, define the function $T(f) \in C(I_\ell)$ by

$$T(f)(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt \text{ for all } x \in I_\ell.$$

Theorem 10.3.C (continued 2)

Proof (continued). A solution of the integral equation (15) is a fixed point of the mapping $T : X_\ell \to C(I_\ell)$. Since $C(I_\ell)$ is a complete metric space by Proposition 9.10 (where the norm on $C(I_\ell)$ has the max norm) and X_ℓ is a closed subset of $C(I_\ell)$ by Proposition 9.11, then X_ℓ is a complete metric space by Proposition 9.12(iii). We will show that if ℓ is chosen sufficiently small, then $T(X_\ell) \subseteq X_\ell$ and $T : X_\ell \to X_\ell$ is a contract. The Banach Contraction Principle will then imply that $T : X_\ell \to X_\ell$ has a unique fixed point, and hence integral equation (15) and differential equation (14) have unique solutions.

Since rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ is compact (by the Heine-Borel Theorem) and g is continuous by hypothesis, then g is the Extreme Value Theorem (Proposition 9.22) there is a positive number K such that $|g(x, y)| \leq K$ for all $(x, y) \in R$.

Theorem 10.3.C (continued 2)

Proof (continued). A solution of the integral equation (15) is a fixed point of the mapping $T : X_\ell \to C(I_\ell)$. Since $C(I_\ell)$ is a complete metric space by Proposition 9.10 (where the norm on $C(I_\ell)$ has the max norm) and X_ℓ is a closed subset of $C(I_\ell)$ by Proposition 9.11, then X_ℓ is a complete metric space by Proposition 9.12(iii). We will show that if ℓ is chosen sufficiently small, then $T(X_\ell) \subseteq X_\ell$ and $T : X_\ell \to X_\ell$ is a contract. The Banach Contraction Principle will then imply that $T : X_\ell \to X_\ell$ has a unique fixed point, and hence integral equation (15) and differential equation (14) have unique solutions.

Since rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ is compact (by the Heine-Borel Theorem) and g is continuous by hypothesis, then g is the Extreme Value Theorem (Proposition 9.22) there is a positive number K such that $|g(x, y)| \leq K$ for all $(x, y) \in R$.

Theorem 10.3.C (continued 3)

Proof (continued). For $f \in X_\ell$ and $x \in I_\ell$ we have, by the definition of T,

$$|t(f)(x) - y_0| = \left| \int_{x_0}^{x} g(t, f(t))\, dt \right| \leq \ell K,$$

so that $T(X_\ell) \subseteq X_\ell$ if $\ell K \leq b$. Observe that for functions $f_1, f_2 \in X_\ell$ and $x \in I_\ell$, by hypothesized inequality (16) we have
$|g(x, f_1(x)) - g(x, f_2(x))| \leq M\, \rho_{\max}(f_1, f_2)$ since
$|y_1 - y_2| = |f(x_1) - f(x_2)| \leq \rho_{\max}(f_1, f_2)$. So by the linearity and monotonicity of integrals, we have

$$\begin{aligned} |T(f_1)(x) - T(f_2)(x)| &= \left| \int_{x_0}^{x} [g(t, f_1(t)) - g(t, f_2(t))]\, dt \right| \\ &\leq |x - x_0| M \rho_{\max}(f_1, f_2) \text{ by the previous inequality} \\ &\leq \ell M \rho_{\max}(f_1, f_2) \text{ since } x \in I_\ell = [x_0 - \ell, x_0 + \ell]. \end{aligned}$$

This inequality, together with the inclusion $T(X_\ell) \subseteq X_\ell$ if $\ell K \leq b$, implies that $T : X_\ell \to X_\ell$ is a contraction if $\ell K \leq b$ and $\ell M < 1$.

Theorem 10.3.C (continued 3)

Proof (continued). For $f \in X_\ell$ and $x \in I_\ell$ we have, by the definition of T,

$$|t(f)(x) - y_0| = \left| \int_{x_0}^{x} g(t, f(t))\, dt \right| \leq \ell K,$$

so that $T(X_\ell) \subseteq X_\ell$ if $\ell K \leq b$. Observe that for functions $f_1, f_2 \in X_\ell$ and $x \in I_\ell$, by hypothesized inequality (16) we have $|g(x, f_1(x)) - g(x, f_2(x))| \leq M\, \rho_{\max}(f_1, f_2)$ since $|y_1 - y_2| = |f(x_1) - f(x_2)| \leq \rho_{\max}(f_1, f_2)$. So by the linearity and monotonicity of integrals, we have

$$\begin{aligned} |T(f_1)(x) - T(f_2)(x)| &= \left| \int_{x_0}^{x} [g(t, f_1(t)) - g(t, f_2(t))]\, dt \right| \\ &\leq |x - x_0| M \rho_{\max}(f_1, f_2) \text{ by the previous inequality} \\ &\leq \ell M \rho_{\max}(f_1, f_2) \text{ since } x \in I_\ell = [x_0 - \ell, x_0 + \ell]. \end{aligned}$$

This inequality, together with the inclusion $T(X_\ell) \subseteq X_\ell$ if $\ell K \leq b$, implies that $T : X_\ell \to X_\ell$ is a contraction if $\ell K \leq b$ and $\ell M < 1$.

Theorem 10.3.C (continued 4)

Theorem 10.3.C. The Picard Local Existence Theorem.
Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ containing the point (x_0, y_0). Suppose the function $g : \mathcal{O} \to \mathbb{R}^2$ is continuous and there is a positive number M for which the following Lipschitz property in the second variable holds uniformly with respect to the first variable:

$$|g(x, y_1)-g(x, y_2)| \leq M|y_1-y_2| \text{ for all points } (x, y_1) \text{ and } (x, y_2) \text{ in } \mathcal{O}. \quad (16)$$

Then there is an open interval I containing x_0 on which the following differential equation has a unique solution:

$$\begin{aligned} f'(x) &= g(x, f(x)) \text{ for all } x \in I \\ f(x_0) &= y_0. \end{aligned} \quad (14)$$

Proof (continued). So we define $\ell = \min\{b/K, 1/(2M)\}$. Then the Banach Contraction Principle applied to $T : X_\ell \to X_\ell$ implies that T has a unique fixed point and, as described above, the integral equation (15) and the differential equation (14) each have a unique solution, as claimed. □