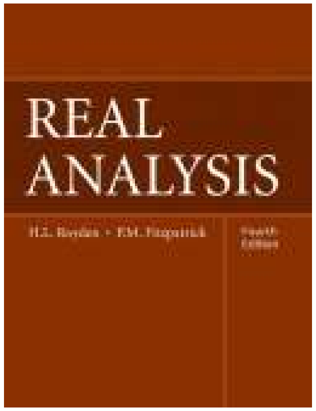
Real Analysis

Chapter 11. Topological Spaces: General Properties
11.1. Open Sets, Closed Sets, Bases, and Subbases—Proofs of Theorems



Proposition 11.1

Proposition 11.1. A subset E of a topological space X is open if and only if for each point $x \in E$ there is a neighborhood of x that is contained in E.

Proof. Let E be open. Then for each $x \in E$, E itself is a neighborhood of x that is contained in E.

Let each point $x \in E$ be contained in a neighborhood of x that is contained in E, say $x \in E_x$ where E_x is such a neighborhood. Then $E = \cup_{x \in E} E_x$ and so by property (iii), E is open. □

Proposition 11.2

Proposition 11.2. For a nonempty set X, let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a base for a topology for X if and only if

(i) $\mathcal{B}$ covers X (that is, $X = \cup_{B \in \mathcal{B}} B$).

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a set $B \in \mathcal{B}$ for which $x \in B \subset B_1 \cap B_2$.

The unique topology that has $\mathcal{B}$ as its base consists of $\varnothing$ and unions of subcollections of $\mathcal{B}$.

Proof. Let collection $\mathcal{B}$ of subsets of X satisfy properties (i) and (ii). Define $\mathcal{T}$ to be the collection of unions of subcollections of $\mathcal{B}$ together with $\varnothing$. Since $X = \cup_{B \in \mathcal{B}} B$ by (i), then $X \in \mathcal{T}$. If $\{\mathcal{O}_i\} \subset \mathcal{T}$ is any collection of sets in $\mathcal{T}$, then we have that for all i, $\mathcal{O}_i = \cup B_{i,j}$ for some $B_{i,j} \in \mathcal{B}$. So $\cup_i \mathcal{O}_i = \cup_{i,j} B_{i,j}$ and hence $\cup_i \mathcal{O}_i \in \mathcal{T}$. Finally, let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{T}$. If $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$ then $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{T}$ by the definition of $\mathcal{T}$.

Proposition 11.2 (continued)

Proof (continued). If $x \in \mathcal{O}_1 \cap \mathcal{O}_2$, then $x \in B_1 \subset \mathcal{O}_1$ and $x \in B_2 \subset \mathcal{O}_2$ for some $B_1, B_2 \in \mathcal{B}$ (since every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$). Then $x \in B_1 \cap B_2$ and so by (ii), there is $B_x \in \mathcal{B}$ with $x \in B_x \subset B_1 \cap B_2$. This holds for each $x \in \mathcal{O}_1 \cap \mathcal{O}_2$, so $\mathcal{O}_1 \cap \mathcal{O}_2 = \cup_{x \in \mathcal{O}_1 \cap \mathcal{O}_2} B_x$ and so $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{T}$. Therefore, $\mathcal{T}$ is a topology.

Let $x \in X$ and let U be a neighborhood of x in $\mathcal{T}$. Then, as above, $x \in B_x \subset U$ for some $B_x \in \mathcal{B}$ and so $\mathcal{B}$ is a base for topology $\mathcal{T}$. Since by definition, a base for a topology is a collection of open sets, since by property (ii) of the definition of topology, a topology generated by $\mathcal{B}$ must contain all unions of subcollections of $\mathcal{B}$. In addition, if U is an open set in the topology with $\mathcal{B}$ as a base, then (as argued above) $U = \cup_{x \in U} B_x$ for some $B_x \in \mathcal{B}$ (where B_x exists by the definition of base of a topology). So the topology generated by $\mathcal{B}$ is unique.

The converse holds by Problem 11.3 □

Proposition 11.3

Proposition 11.3. For E a subset of a topological space $(X, \mathcal{T})$, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Proof. The set $\overline{E}$ is closed provided it contains all of its points of closure (that is, $\overline{(\overline{E})} = \overline{E}$). Let x be a point of closure of $\overline{E}$. Consider a neighborhood U_x of x. By the definition of "point of closure of $\overline{E}$," there is a point $x' \in \overline{E} \cap U_x$. Since x' is a point of closure of E (because $x' \in \overline{E}$) and U_x is a neighborhood of x', then there is a point $x'' \in E \cap U_x$ (again, by the definition of closure). Therefore every neighborhood of x contains a point of E and hence $x \in \overline{E}$. So $\overline{(\overline{E})} \subset \overline{E}$ and, of course, $\overline{E} \subset \overline{(\overline{E})}$. So $\overline{E}$ is closed.

By the definition of "point of closure," if $A \subset B$ then $\overline{A} \subset \overline{B}$. So if F is closed and $E \subset F$ then $E \subset \overline{E} \subset \overline{F} = F$ and so $\overline{E}$ is the smallest closed set containing E. □

Proposition 11.4

Proposition 11.4. A subset of a topological space $(X, \mathcal{T})$ is open if and only if its complement in X is closed.

Proof. Suppose E is open in X. Let $x \in \overline{X \sim E}$. ASSUME $x \in E$. Then there is a neighborhood of x contained in E, but this neighborhood of x does not intersect $X \sim E$ and then x is not in the closure of $X \sim E$, a CONTRADICTION. So $x \notin E$. This $x \in X \sim E$ and so $\overline{X \sim E} = X \sim E$ and $X \sim E$ is closed.

Suppose $X \sim E$ is closed. Let $x \in E$. Then $x \notin X \sim E$. Let U be any neighborhood of x. ASSUME U is not a subset of E. Then $U \cap (X \sim E) \neq \varnothing$ and it follows that $x \in \overline{X \sim E}$. But since $X \sim E$ is closed, then $x \in \overline{X \sim E} = C \sim E$, a CONTRADICTION. So the assumption that every neighborhood of x is not a subset of E is false. That is, some neighborhood of x is a subset of E and E is open. □