Real Analysis

Chapter 11. Topological Spaces: General Properties

11.1. Open Sets, Closed Sets, Bases, and Subbases—Proofs of Theorems

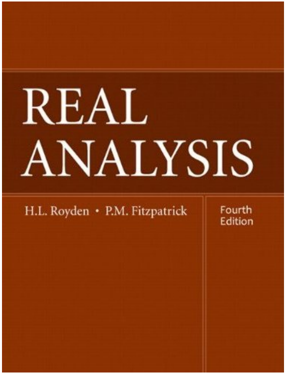

Table of contents

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Proposition 11.2

Proposition 11.2. For a nonempty set X, let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a base for a topology for X if and only if

(i) $\mathcal{B}$ covers X (that is, $X = \cup_{B \in \mathcal{B}} B$).

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a set $B \in \mathcal{B}$ for which $x \in B \subset B_1 \cap B_2$.

The unique topology that has $\mathcal{B}$ as its base consists of $\varnothing$ and unions of subcollections of $\mathcal{B}$.

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Let $x \in X$ and let U be a neighborhood of x in $\mathcal{T}$. Then, as above, $x \in B_x \subset U$ for some $B_x \in \mathcal{B}$ and so $\mathcal{B}$ is a base for topology $\mathcal{T}$. Since by definition, a base for a topology is a collection of open sets, since by property (ii) of the definition of topology, a topology generated by $\mathcal{B}$ must contain all unions of subcollections of $\mathcal{B}$.

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The converse holds by Problem 11.3 □

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Proposition 11.3. For E a subset of a topological space $(X, \mathcal{T})$, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Proof. The set $\overline{E}$ is closed provided it contains all of its points of closure (that is, $\overline{(\overline{E})} = \overline{E}$). Let x be a point of closure of $\overline{E}$. Consider a neighborhood U_x of x.

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By the definition of "point of closure," if $A \subset B$ then $\overline{A} \subset \overline{B}$. So if F is closed and $E \subset F$ then $E \subset \overline{E} \subset \overline{F} = F$ and so $\overline{E}$ is the smallest closed set containing E. □

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Suppose $X \sim E$ is closed. Let $x \in E$. Then $x \notin X \sim E$. Let U be any neighborhood of x.

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Suppose $X \sim E$ is closed. Let $x \in E$. Then $x \notin X \sim E$. Let U be any neighborhood of x. ASSUME U is not a subset of E. Then $U \cap (X \sim E) \neq \varnothing$ and it follows that $x \in \overline{X \sim E}$. But since $X \sim E$ is closed, then $x \in \overline{X \sim E} = C \sim E$, a CONTRADICTION.

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