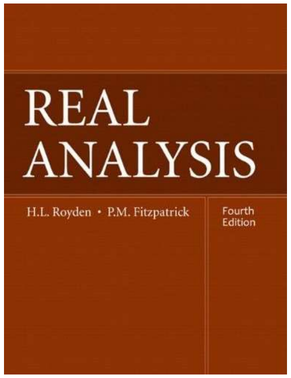
Real Analysis

Chapter 11. Topological Spaces: General Properties
11.2. The Seperation Properties—Proofs of Theorems



Proposition 11.6

Proposition 11.6. A topological space $(X, \mathcal{T})$ is a Tychonoff space if and only if every set consisting of a single point is closed.

Proof. Let $x \in X$. Set $\{x\}$ is closed if and only if $X \sim \{x\}$ is open by Proposition 11.4. Now $X \sim \{x\}$ is open if and only if for each $y \in X \sim \{x\}$ there is a neighborhood of y that is contained in $X \sim \{x\}$, by Proposition 11.1; that is, if and only if there is a neighborhood of y that does not contain x. So $(X, \mathcal{T})$ is Tychonoff (and any two points can be separated in the Tychonoff sense) if and only if singletons form closed sets. □

Proposition 11.7

Proposition 11.7. Every metric space is normal.

Proof. Let$(X, \mathcal{T})$ be a metric space. Define the distance between a subset $F \subset X$ and a point $x \in X$ as $\text{dist}(x, F) = \in \{\rho(x, x') \mid x' \in F\}$. Let F_1 and F_2 be closed disjoint subsets of X. Define

$$\mathcal{O}_1 = \{x \in X \mid \text{dist}(X, F_1) < \text{dist}(x, F_2)\}$$

and

$$\mathcal{O}_2 = \{x \in X \mid \text{dist}(X, F_2) < \text{dist}(x, F_1)\}.$$

Since the complement of a closed set is open by Proposition 11.4, then for closed F and $x \notin F$ we have $\text{dist}(x, F) > 0$ (that is, $x \in X \sim F$ and there is an ε neighborhood of x contained in $X \sim F$ since $X \sim F$ is open). So $F_1 \subset \mathcal{O}_1$ and $F_2 \subset \mathcal{O}_2$. Also, $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$ (or else there is some x such that both $\text{dist}(x, F_1) < \text{dist}(x, F_2)$ AND $\text{dist}(x, F_2) < \text{dist}(x, F_1)$). In addition, $\mathcal{O}_1$ and $\mathcal{O}_2$ are open based on the triangle inequality for ρ (see page 183). □

Proposition 11.8

Proposition 11.8. Let $(X, \mathcal{T})$ be a Tychonoff topological space. Then X is normal if and only if whenever $\mathcal{U}$ is a neighborhood of a closed subset F of X, then there is another neighborhood of F whose closure is contained in $\mathcal{U}$; that is, there is an open $\mathcal{O}$ for which $F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}$.

Proof. Suppose X is normal. Let F be closed and let $\mathcal{U}$ be a neighborhood of F. So F and $X \sim \mathcal{U}$ are disjoint closed sets. Hence, there are disjoint open sets $\mathcal{O}$ and $\mathcal{V}$ for which $F \subset \mathcal{O}$ and $X \sim \mathcal{U} \subset \mathcal{V}$. Thus $\mathcal{O} \subset X \sim \mathcal{V}$ (since $\mathcal{O} \cap \mathcal{V} = \varnothing$) and so $\mathcal{O} \subset X \sim \mathcal{V} \subset \mathcal{U}$ (since $X \sim \mathcal{U} \subset \mathcal{V}$). Since $\mathcal{O} \subset X \sim \mathcal{V}$ and $X \sim \mathcal{V}$ is closed, then $\overline{\mathcal{O}} \subset \overline{X \sim \mathcal{V}} = X \sim \mathcal{V} \subset \mathcal{U}$.

Suppose set $\mathcal{O}$ exists as claimed. Let A and B be disjoint closed subsets of X. Then $A \subset X \sim B$ an d$X \sim B$ is open by Proposition 11.4. Thus there is open set $\mathcal{O}$ for which $A \subset \mathcal{O} \subset \overline{\mathcal{O}} \subset X \sim B$ by hypothesis. Then $\mathcal{O}$ and $X \sim \overline{\mathcal{O}}$ are disjoint with $A \subset \mathcal{O}$ and $B \subset X \sim \overline{\mathcal{O}}$. So the space is normal. □