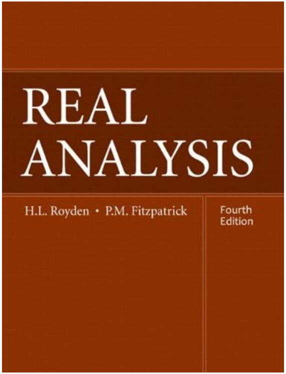
Real Analysis

Chapter 11. Topological Spaces: General Properties

11.2. The Seperation Properties—Proofs of Theorems



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Proposition 11.7

Proposition 11.7. Every metric space is normal.

Proof. Let$(X, \mathcal{T})$ be a metric space. Define the distance between a subset $F \subset X$ and a point $x \in X$ as $\text{dist}(x, F) =\in \{\rho(x, x') \mid x' \in F\}$. Let F_1 and F_2 be closed disjoint subsets of X.

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$$\mathcal{O}_1 = \{x \in X \mid \text{dist}(X, F_1) < \text{dist}(x, F_2)\}$$

and

$$\mathcal{O}_2 = \{x \in X \mid \text{dist}(X, F_2) < \text{dist}(x, F_1)\}.$$

Since the complement of a closed set is open by Proposition 11.4, then for closed F and $x \notin F$ we have $\text{dist}(x, F) > 0$ (that is, $x \in X \sim F$ and there is an ε neighborhood of x contained in $X \sim F$ since $X \sim F$ is open). So $F_1 \subset \mathcal{O}_1$ and $F_2 \subset \mathcal{O}_2$.

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Proposition 11.8

Proposition 11.8. Let $(X, \mathcal{T})$ be a Tychonoff topological space. Then X is normal if and only if whenever $\mathcal{U}$ is a neighborhood of a closed subset F of X, then there is another neighborhood of F whose closure is contained in $\mathcal{U}$; that is, there is an open $\mathcal{O}$ for which $F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}$.

Proof. Suppose X is normal. Let F be closed and let $\mathcal{U}$ be a neighborhood of F. So F and $X \sim \mathcal{U}$ are disjoint closed sets. Hence, there are disjoint open sets $\mathcal{O}$ and $\mathcal{V}$ for which $F \subset \mathcal{O}$ and $X \sim \mathcal{U} \subset \mathcal{V}$.

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