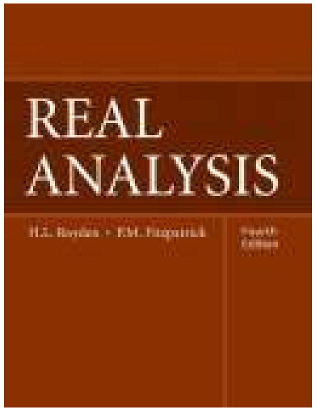
Real Analysis

Chapter 11. Topological Spaces: General Properties

11.3. Countability and Separation—Proofs of Theorems



Proposition 11.9

Proposition 11.9. Let $(X, \mathcal{T})$ be a first countable topological space. For a subset E of X, a point $x \in X$ is a point of closure of E if and only if x is a limit point of a sequence in E. Therefore, a subset E of X is closed if and only if whenever a sequence in E converges to $x \in X$, the point x belongs to E.

Proof. Suppose x is a limit point of a sequence $\{x_n\}$ in E. Let $\mathcal{U}$ be an open set containing x. Then there is $N \in \mathbb{N}$ such that if $n \geq N$, then $x_n \in \mathcal{U}$. So $\mathcal{U}$ contains a point in E and hence x is a point of closure of E.

Suppose x is a point of closure in E. Since $(X, \mathcal{T})$ is first countable, there is a base at x, say $\{B_i\}_{i=1}^{\infty}$. Define $C_n = \cap_{i=1}^{n} B_i$. Since $x \in B_i$ for all i then $C_n \neq \varnothing$ Also, each C_n is open. Choose $x_n \in C_n$ to create sequence $\{x_n\}$. Let $\mathcal{U}$ be a neighborhood of x. Then $B_N \subset \mathcal{U}$ for some $N \in \mathbb{N}$ (by the definition of base at x). Notice that $C_n \subset B_N$ for all $n \geq N$. So $x_n \in \mathcal{U}$ for all $n \geq N$ and x is a limit of sequence $\{x_n\}$. □