Real Analysis

Chapter 11. Topological Spaces: General Properties

11.3. Countability and Separation—Proofs of Theorems

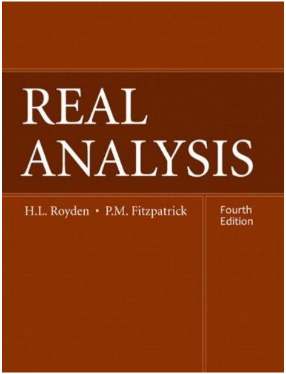

Table of contents

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Proof. Suppose x is a limit point of a sequence $\{x_n\}$ in E. Let $\mathcal{U}$ be an open set containing x. Then there is $N \in \mathbb{N}$ such that if $n \geq N$, then $x_n \in \mathcal{U}$. So $\mathcal{U}$ contains a point in E and hence x is a point of closure of E.

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Suppose x is a point of closure in E. Since $(X, \mathcal{T})$ is first countable, there is a base at x, say $\{B_i\}_{i=1}^{\infty}$. Define $C_n = \cap_{i=1}^{n} B_i$. Since $x \in B_i$ for all i then $C_n \neq \varnothing$ Also, each C_n is open.

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