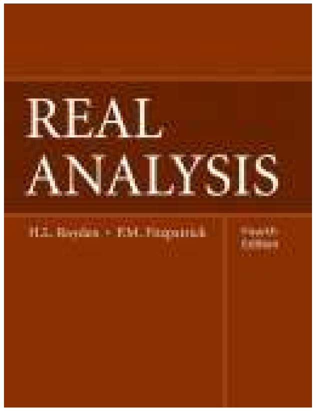
Real Analysis

Chapter 11. Topological Spaces: General Properties
11.4. Continuous Mappings Between Topological Spaces—Proofs



Proposition 11.10

Proposition 11.10. A mapping $f : X \to Y$ between topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ is continuous if and only if for any subset $\mathcal{O} \in \mathcal{S}$, its inverse image under f, $f^{-1}(\mathcal{O}) \in \mathcal{T}$.

Proof. Suppose f is continuous. Let $\mathcal{O} \in \mathcal{S}$. By Proposition 11.1, to show that $f^{-1}(\mathcal{O})$ is open it suffices to show that each point in $f^{-1}(\mathcal{O})$ has a neighborhood that is contained in $f^{-1}(\mathcal{O})$. Let $x \in f^{-1}(\mathcal{O})$. Then by the continuity of f at x there is a neighborhood of x that is mapped into $\mathcal{O}$. Therefore this neighborhood of x is contained in $f^{-1}(\mathcal{O})$. Hence $f^{-1}(\mathcal{O}) \in \mathcal{T}$.

Conversely, if f^{-1} maps open sets to open sets, then for any neighborhood of $f(x_0)$, there is a neighborhood $\mathcal{U}$ of x_0 for which $f(\mathcal{U}) \subset \mathcal{O}$; namely, $\mathcal{U} = f^{-1}(\mathcal{O})$. So f is continuous at x_0. □

Proposition 11.13

Proposition 11.13. Let X be a nonempty set and $\mathcal{F} = \{f_\lambda : X \to X_\lambda\}_{\lambda \in \Lambda}$ a collection of mappings where each X_λ is a topological space. The weak topology for X induced by $\mathcal{F}$ is the topology on X that has the fewest number of sets among the topologies on X for which each mapping $f_\lambda : X \to X_\lambda$ is continuous.

Proof. By Proposition 11.10, for each $\lambda \in \Lambda$, $f_\lambda : X \to X_\lambda$ is continuous if and only if the inverse image under f_λ of each open set in X_λ is open in X. The weak topology includes $\mathcal{F} = \{f_\alpha^{-1}(\mathcal{O}_\alpha) \mid f_\alpha \in \mathcal{F}, \mathcal{O}_\alpha \text{ is open in } X_\alpha\}$ and so the weak topology, be definition, has all inverse images of open sets open. So each $f_\lambda : X \to X_\lambda$ is continuous in the weak topology. By definition, the weak topology has the fewest number of sets among all topologies with this property. □