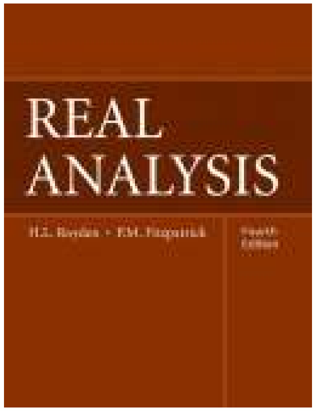
Real Analysis

Chapter 11. Topological Spaces: General Properties
11.5. Compact Topological Spaces—Proofs of Theorems



Proposition 11.14

Proposition 11.14. A topological space $(X, \mathcal{T})$ is compact if and only if every collection of closed subsets of X that possesses the finite intersection property has nonempty intersection.

Proof. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X. Define $C_\lambda = X \sim \mathcal{O}_\lambda$. Then each C_λ is closed. Also, by DeMorgan's Laws,

$$X = \cup_{\lambda \in \Lambda} \mathcal{O}_\lambda \text{ implies } \varnothing = \cap_{\lambda \in \Lambda} C_\lambda.$$

Conversely, if $\{C_\lambda\}_{\lambda \in \Lambda}$ is a collection of closed sets and we define $\mathcal{O}_\lambda = X \sim C_\lambda$, then by DeMorgan's Laws,

$$\varnothing = \cap_{\lambda \in \Lambda} C_\lambda \text{ implies } X = \cup_{\lambda \in \Lambda} \mathcal{O}_\lambda.$$

Suppose X is compact and let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a collection of closed sets with the finite intersection property. ASSUME $\cap_{\lambda \in \Lambda} C_\lambda = \varnothing$. Define $\mathcal{O}_\lambda$ as above.

Proposition 11.14 (continued)

Proof (continued). Then $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is an open cover of X. Since X is compact, then for some $\lambda = 1, 2, \ldots, n$ we have $X \subseteq \cup_{i=1}^n \mathcal{O}_i$. So $\cap_{i=1}^n C_i = \varnothing$, but this contradicts the fact that $\{C_\lambda\}_{\lambda \in \Lambda}$ has the finite intersection property. So the assumption that $\cap_{\lambda \in \Lambda} C_\lambda = \varnothing$ is false and it must be that $\cap_{\lambda \in \Lambda} C_\lambda \neq \varnothing$. That is, every collection of closed sets with the finite intersection property has nonempty intersection.

Suppose every collection of closed sets with the finite intersection property has nonempty intersection. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X. ASSUME X is not compact. Then no finite subcollection of $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is a cover of X. Define $C_\lambda = X \sim \mathcal{O}_\lambda$. Then any finite subcollection of $\{C_\lambda\}_{\lambda \in \Lambda}$ is nonempty (and, of course, each C_λ is closed). So, by hypotheses, $\cap_{\lambda \in \Lambda} C_\lambda \neq \varnothing$. But then $\cup_{\lambda \in \Lambda} \mathcal{O}_\lambda \neq X$ and $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is not a cover of X, a CONTRADICTION. So the assumption that X is not compact is false. □

Proposition 11.15

Proposition 11.15. A closed subset of a compact topological space $(X, \mathcal{T})$ is compact.

Proof. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of K with open sets in $\mathcal{T}$. Since $X \sim K$ is open in $\mathcal{T}$, then $\{X \sim K\} \cup \{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is an open cover of X. Since X is compact, there is a finite subcover of X. This subcover of X is also a cover of K. If $X \sim K$ is included in the subcover of K, then it can be omitted from the subcover of K, since it is disjoint from K. The resulting subcover of K is a subset of the original cover of K and hence K is compact. □

Proposition 11.16

Proposition 11.16. A compact subspace K of a Hausdorff topological space $(X, \mathcal{T})$ is a closed subset of K.

Proof. If $K = X$, then K is closed. Otherwise, let $y \in X \sim K$. Since X is Hausdorff, for each $x \in K$ there are disjoint neighborhoods $\mathcal{O}_x$ and $\mathcal{U}_x$ of x and y respectively. Then $\{\mathcal{O}_x\}_{x \in K}$ is an open cover of K. Since K is compact, there is a finite subcover $\{\mathcal{O}_{x_1}, \mathcal{O}_{x_2}, \ldots, \mathcal{O}_{x_n}\}$. Define $\mathcal{N} = \cap_{i=1}^n \mathcal{U}_{x_i}$. Then $\mathcal{N}$ is a neighborhood of y which is disjoint from each $\mathcal{O}_{x_i}$ (since $\mathcal{O}_{x_i}$ and $\mathcal{U}_{x_i}$ are disjoint). Hence $\mathcal{N} \subset X \sim K$ since the $\{\mathcal{O}_{x_i}\}_{i=1}^n$ cover K. Since $y \in X \sim K$ is arbitrary, then $X \sim K$ is open and so K is closed. □

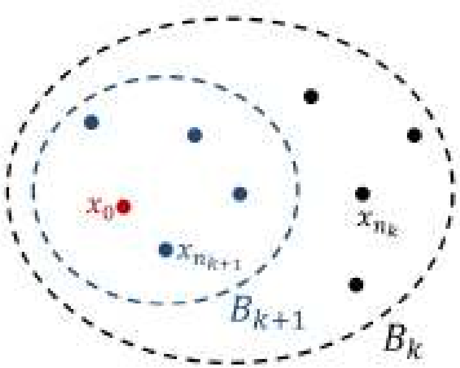
Proposition 11.17

Proposition 11.17. Let $(X, \mathcal{T})$ be a second countable topological space. Then $(X, \mathcal{T})$ is compact if and only if it is sequentially compact.

Proof. Let $(X, \mathcal{T})$ be compact. Let $\{x_n\}$ be a sequence in X. For each $n \in \mathbb{N}$, let F_n be the closure of $\{x_k \mid k \geq n\}$. Then $\{F_n\}$ is a decreasing sequence of nonempty closed sets. So $\{F_n\}$ has the finite intersection property. By Proposition 11.14, $\cap_{n=1}^\infty F_n \neq \varnothing$, so choose $x_0 \in \cap_{n=1}^\infty F_n$. Since X is second countable, it is first countable and so has a base $\{B_n\}_{n=1}^\infty$ for the topology at point x_0. Without loss of generality we may assume $B_{n+1} \subset B_n$ (or else we could replace B_n with $\cap_{k=1}^n B_n$ and this then produces a base at x with this decreasing property). Since $x_0 \in F_n$ for all $n \in \mathbb{N}$, then x_0 is a point of closure of $\{x_k \mid k \geq n\}$ for all $n \in \mathbb{N}$. So neighborhood B_n of x_0 has nonempty intersection with $\{x_k \mid k \geq n\}$ for all $n \in \mathbb{N}$. So we can inductively choose x_{n_k} (where the sequence of subscripts n_k is strictly increasing) in B_k.

Proposition 11.17 (continued 1)

Proof (continued).



Since for each neighborhood $\mathcal{O}$ of x_0, there is an index N for which $B_n \subset \mathcal{O}$ for $n \geq N$ (by the definition of base at x_0 and the nestedness of the B_n's), the subsequence $\{x_{n_k}\}$ converges to x_0. So X is sequentially compact.

Proposition 11.17 (continued 2)

Proof (continued). Suppose X is sequentially compact. Since X is second countable, every open cover has a countable subcover (by the definition of 2nd countable). So, to show that X is compact it suffices to show that every countable open cover of X has a finite subcover. Let $\{\mathcal{O}_n\}_{n=1}^\infty$ be such a cover. ASSUME there is no finite subcover. Then for each $n \in \mathbb{N}$, there is an index $m(n) > n$ for which $\mathcal{O}_{m(n)} \sim (\cup_{i=1}^n \mathcal{O}_i) \neq \varnothing$ (or else $\{\mathcal{O}_i\}_{i=1}^n$ is a finite subcover of X). So for each $n \in \mathbb{N}$, choose $x_n \in \mathcal{O}_{m(n)} \sum (\cup_{i=1}^n \mathcal{O}_i)$. Then since X is sequentially compact, a subsequence of $\{x_n\}$ converges to some $x_0 \in X$. But $\{\mathcal{O}_n\}_{n=1}^\infty$ is an open cover of X, so there is some $\mathcal{O}_N$ that is a neighborhood of x_0. Therefore, there are infinitely many indices n for which x_n belongs to $\mathcal{O}_N$ (these terms being in the subsequence of $\{x_n\}$ which converges to x_0). But by the construction of $\{x_n\}$, $x_n \notin \mathcal{O}_N$ for $n > N$. So this CONTRADICTION shows that the assumption that X is not compact is false. □

Theorem 11.18

Theorem 11.18. A compact Hausdorff space is normal.

Proof. Let $(X, \mathcal{T})$ be compact and Hausdorff. Let F be a closed subset of X and let point $x \in X \sim F$. Since $(X, \mathcal{T})$ is Hausdorff, for each $y \in F$ there are disjoint neighborhoods $\mathcal{O}_x$ and $\mathcal{U}_y$ of x and y, respectively. Then $\{\mathcal{U}_y\}_{y \in F}$ is an open cover of F. But F is a closed subset of a compact space and so by Proposition 11.15 is itself compact. So there is a finite subcover $\{\mathcal{U}_{y_1}, \mathcal{U}_{y_2}, \ldots, \mathcal{U}_{y_n}\}$ of F. Define $\mathcal{N} = \cap_{i=1}^n \mathcal{O}_{y_i}$. So $\mathcal{N}$ is open and $F \subset \mathcal{N}$ is disjoint from $\cup_{i=1}^n \mathcal{U}_{y_i}$, a neighborhood of F. Thus $(X, \mathcal{T})$ is regular.
Let F and G be disjoint closed sets. Since $(X, \mathcal{T})$ is regular, for each $g \in G$ there are disjoint $\mathcal{V}_g$ and $\mathcal{W}_g$ such that $F \subset \mathcal{V}_g$ and $g \in \mathcal{W}_g$. Then $\{\mathcal{V}_g\}_{g \in G}$ is an open cover of F. By Proposition 11.15, F is compact and so there is some finite $\{\mathcal{V}_{g_i}\}_{i=1}^m$ subcover of F. Then (similar to above) the open sets $\cup_{i=1}^m \mathcal{V}_{g_i}$ and $\cap_{i=1}^m \mathcal{W}_{g_i}$ separates F and G. Therefore $(X, \mathcal{T})$ is normal. □

Proposition 11.20

Proposition 11.20. The continuous image of a compact topological space is compact.

Proof. Let f be a continuous mapping of $(X, \mathcal{T})$ to $(Y, \mathcal{S})$. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open covering of $f(X)$. Then since f is continuous, by Proposition 11.10, $\{f^{-1}(\mathcal{O}_\lambda)\}_{\lambda \in \Lambda}$ is an open cover of X. Since X is compact, there is a finite subcover $\{f^{-1}(\mathcal{O}_{\lambda_i})\}_{i=1}^n$ of X. Then the finite collection $\{\mathcal{O}_{\lambda_i}\}_{i=1}^n$ is a cover of $f(X)$ and $f(X)$ is compact. □

Corollary 11.21

Corollary 11.21. A continuous real-valued function on a compact topological space takes a maximum and minimum functional value.

Proof. Let $(X, \mathcal{T})$ be compact $f : X \to \mathbb{R}$ be continuous. By Proposition 11.20, $f(X)$ is a compact set of real numbers. So, by the Heine-Borel Theorem, $f(X)$ is closed and bounded. So f attains a maximum and minimum value (namely, $\sup f(X) = \max f(X)$ and $\inf f(X) = \min f(X)$). □

Proposition 11.19

Proposition 11.19. A continuous one to one mapping f of a compact space $(X, \mathcal{T})$ onto a Hausdorff space Y is a homeomorphism.

Proof. Since f is given to be continuous, one to one, and onto then we need only show that f^{-1} is continuous. This can be done by showing f maps open sets to closed sets or, equivalently, f maps closed sets to closed sets. Let F be a closed subset of X. Then F is compact by Proposition 11.15. By Proposition 11.20, $f(F)$ is compact in $(Y, \mathcal{S})$. Hence by Proposition 11.16, since Y is Hausdorff, $f(F)$ is closed. Therefore f^{-1} is continuous and f is a homeomorphism. □