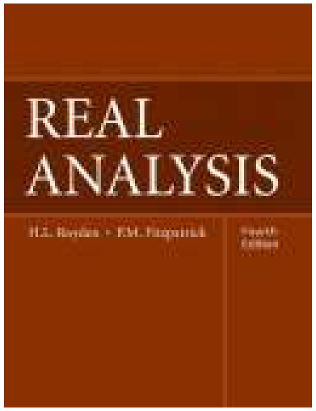
Real Analysis

Chapter 11. Topological Spaces: General Properties
11.6. Connected Topological Spaces—Proofs of Theorems



Proposition 11.22

Proposition 11.22. Let f be a continuous mapping of a connected space $(X, \mathcal{T})$ to a topological space $(Y, \mathcal{S})$. Then its image $f(X)$ is connected.

Proof. Consider the subspace of $(Y, \mathcal{S})$ given by $f(X)$ with the subspace topology inherited from Y. ASSUME $f(X)$ is not connected. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be a separation of $f(X)$. Then $f^{-1}(\mathcal{O}_1)$ and $f^{-1}(\mathcal{O}_2)$ are disjoint (since $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$) nonempty (since f is onto $f(X)$) open sets (by Proposition 11.10) in X whose union is X (since $f(X) = \mathcal{O}_1 \cup \mathcal{O}_2$). Thus this pair $f^{-1}(\mathcal{O}_1)$, $f^{-1}(\mathcal{O}_2)$ is a separation of X, CONTRADICTING the hypothesis. So the assumption that $f(X)$ is not connected is false, and $f(X)$ is connected. □

Proposition 11.23

Proposition 11.23. A topological space has the intermediate value property if and only if it is connected.

Proof. Let $(X, \mathcal{T})$ be a connected topological space and let $f : X \to \mathbb{R}$ be continuous. Then by Proposition 11.22, $f(X) \subset \mathbb{R}$ is connected. Since a connected set of real numbers is an interval (or singleton) then $f(X)$ is an interval (or singleton) and by definition, $(X, \mathcal{T})$ has the intermediate value property.

The converse of the above is: "If $(X, \mathcal{T})$ has the intermediate value property then it is connected." We prove the contrapositive. Suppose $(X, \mathcal{T})$ is not connected. Then there is a pair of nonempty disjoint open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $X = \mathcal{O}_1 \cup \mathcal{O}_2$. Define f on X as

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathcal{O}_1 \\ 1 & \text{if } x \in \mathcal{O}_2. \end{cases}$$

Proposition 11.23 (continued)

Proposition 11.23. A topological space has the intermediate value property if and only if it is connected.

Proof (continued). Now $f^{-1}(A)$ is an open subset of X for every subset $A \subset \mathbb{R}$ (namely, either $\mathcal{O}_1$, $\mathcal{O}_2$, or X). In particular, for any open $A \subset \mathbb{R}$, $f^{-1}(A)$ is open in X. So by Proposition 11.10, f is continuous. But $f(X) = \{0, 1\}$ is not an interval in $\mathbb{R}$ and so f does not have the intermediate value property. □