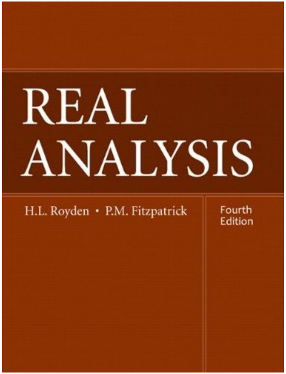
Real Analysis

Chapter 11. Topological Spaces: General Properties

11.6. Connected Topological Spaces—Proofs of Theorems



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Proposition 11.23

Proposition 11.23. A topological space has the intermediate value property if and only if it is connected.

Proof. Let $(X, \mathcal{T})$ be a connected topological space and let $f : X \to \mathbb{R}$ be continuous. Then by Proposition 11.22, $f(X) \subset \mathbb{R}$ is connected. Since a connected set of real numbers is an interval (or singleton) then $f(X)$ is an interval (or singleton) and by definition, $(X, \mathcal{T})$ has the intermediate value property.

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The converse of the above is: "If $(X, \mathcal{T})$ has the intermediate value property then it is connected." We prove the contrapositive. Suppose $(X, \mathcal{T})$ is not connected. Then there is a pair of nonempty disjoint open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $X = \mathcal{O}_1 \cup \mathcal{O}_2$.

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$$f(x) = \begin{cases} 0 & \text{if } x \in \mathcal{O}_1 \\ 1 & \text{if } x \in \mathcal{O}_2. \end{cases}$$

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