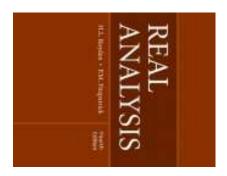# Real Analysis

Chapter 12. Topological Spaces: Three Fundamental Theorems

12.1. Urysohn's Lemma and the Tietze Extension Theorem—Proofs of Theorems

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Lemma 12.2. Let $(X, \mathcal{T})$ be a normal topological space, F a closed subset of X, and $\mathcal{U}$ a neighborhood of F. Then for any open, bounded interval (a, b), there is a dense subset Λ of (a, b) and a normally ascending collection of open subsets of X, $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$, for which

$$F \subseteq \mathcal{O}_\lambda \subseteq \overline{\mathcal{O}}_\lambda \subseteq \mathcal{U} \text{ for all } \lambda \in \Lambda.$$

Proof. Without loss of generality, we take $(a, b) = (0, 1)$ (otherwise we continuously map (a, b) to $(0, 1)$ with $f(x) = (x - a)/(b - a)$ and then apply the result we now prove). For the dense subset of $(0, 1)$ we choose the set of dyadic rationals in $(0, 1)$:

$$\Lambda = \{m/2^n \mid m, n \in \mathbb{N}, 1 \le m \le 2^n - 1\}.$$

Let

$$\Lambda_n = \{m/2^n \mid m \in \mathbb{N}, 1 \le m \le 2^n - 1\}.$$

By Proposition 11.8, there is open $\mathcal{O}_{1/2}$ for which $F \subset \mathcal{O}_{1/2} \subset \overline{\mathcal{O}}_{1/2} \subset \mathcal{U}$.

Lemma 12.2 (continued)

Proof. This is $\mathcal{O}_\lambda$ for all $\lambda \in \Lambda_1 = \{1/2\}$. Again, by Proposition 11.8, with closed F and neighborhood $\mathcal{U} = \mathcal{O}_{1/2}$ of F there is open $\mathcal{O}_{1/4}$ with $F \subset \mathcal{O}_{1/4} \subset \overline{\mathcal{O}}_{1/4} \subset \mathcal{O}_{1/2}$. With closed $\overline{\mathcal{O}}_{1/2}$ and neighborhood $\mathcal{U}$ of $\overline{\mathcal{O}}_{1/2}$ there is by Proposition 11.8 open $\mathcal{O}_{3/4}$ with $\overline{\mathcal{O}}_{1/2} \subset \mathcal{O}_{3/4} \subset \overline{\mathcal{O}}_{3/4} \subset \mathcal{U}$. So we have

$$F \subset \mathcal{O}_{1/4} \subset \overline{\mathcal{O}}_{1/4} \subset \mathcal{O}_{1/2} \subset \overline{\mathcal{O}}_{1/2} \subset \mathcal{O}_{3/4} \subset \overline{\mathcal{O}}_{3/4} \subset \mathcal{U}.$$

So the normally ascending collection $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda_1}$ is extended to normally ascending collection $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda_2}$. We then proceed inductively to define for each $n \in \mathbb{N}$, the normally ascending collection of open sets $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda_n}$. Then $\cup_{n=1}^\infty \{\mathcal{O}_\lambda\}_{\lambda \in \Lambda_n}$ is the desired normally ascending collection of open sets. $\square$

Urysohn's Lemma

Urysohn's Lemma. Let A and B be disjoint closed subsets of a normal topological space $(X, \mathcal{T})$. Then for any closed bounded interval of real numbers $[a, b]$, there is a continuous real-valued function f defined on X that takes values in $[a, b]$, while $f = a$ on A and $f = b$ on B.

Proof. By Lemma 12.2, with $F = A$ and $\mathcal{U} = X \setminus B$, we can choose a dense subset Λ of (a, b) and a normally ascending collection of open subsets of X, $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$, for which $A \subset \mathcal{O}_\lambda \subset X \setminus B$ for all $\lambda \in \Lambda$. Define the function $f : X \to [a, b]$ by setting $f = b$ on $X \setminus \cup_{\lambda \in \Lambda} \mathcal{O}_\lambda$, and otherwise setting $f(x) = \inf\{\lambda \in \Lambda \mid x \in \mathcal{O}_\lambda\}$. Since $A \subset \mathcal{O}_\lambda$ for all λ and Λ is dense in (a, b), then $f(A) = a$. Since $\mathcal{O}_\lambda \subset X \setminus B$ for all λ, $B \cap (\cup_{\lambda \in \Lambda} \mathcal{O}_\lambda) = \varnothing$ and so $f = b$ on B. By Lemma 12.1, f is continuous on X. $\square$

The Tietze Extension Theorem

The Tietze Extension Theorem.

Let $(X, \mathcal{T})$ be a normal topological space, F a closed subset of X, and f a continuous real-valued function on F that takes values in the closed, bounded interval $[a, b]$. Then f has a continuous extension to all of X that also takes values in $[a, b]$.

Proof. Since $[a, b]$ and $[-2, 2]$ are homeomorphic (consider $f : [a, b] \to [-2, 2]$ defined as $f(x) = 4(x - a)/(b - a) - 2)$, we assume without loss of generality that $[a, b] = [-2, 2]$.

We construct a sequence $\{g_n\}$ of continuous real-valued functions on X with the following properties:

(1) For each $n \in \mathbb{N}$, $|g_n(x)| \leq (2/3)^n$ on X, and
(2) for each $n \in \mathbb{N}$, $|f - (g_1 + g_2 + \cdots + g_n)| \leq (2/3)^n$ on F.

With this sequence constructed, define for each $n \in \mathbb{N}$, the real-valued function s_n on X by $s_n(x) = \sum_{k=1}^{n} g_k(x)$ for $x \in X$. By property (1), we see that for each $x \in X$, the sequence $\{g_n(x)\}$ is a Cauchy sequence of real numbers and hence a convergent sequence.

The Tietze Extension Theorem (continued 1)

Proof (continued). So define $g(x) = \lim_{n\to\infty} s_n(x)$ for each $x \in X$.
Since each g_n is continuous on X then, of course, each s_n is continuous.
By property (1), $\{s_n\}$ converges to g uniformly on X (since $\sum_{n=1}^{\infty} (2/3)^n = 2$). Therefore g is continuous on X. By property (2), f is also the (uniform) pointwise limit of $\{s_n\}$ on F, so $f = g$ on F. Notice that for each $x \in X$

$$g(x) = \left| \sum_{k=1}^{\infty} g_n(x) \right| \leq \sum_{k=1}^{\infty} |g_n(x)| \leq \sum_{k=1}^{\infty} (2/3)^n = 2.$$

So $g(x) \in [-2, 2]$ for all $x \in X$. Therefore g is the desired continuous extension of f to X. We now construct the sequence $\{g_n\}$.

The Tietze Extension Theorem (continued 2)

Proof (continued). We claim that for any $a > 0$ and continuous function $h : F \to \mathbb{R}$ for which $|h| \leq a$ on F, there is a continuous function $g : X \to \mathbb{R}$ such that

$$|g| \leq (2/3)a \text{ on } X \text{ and } |j - g| \leq (2/3)a \text{ on } F.$$

We justify this claim by defining

$$A = \{x \in F \mid h(x) \leq 1(1/3)a\} \text{ and } B = x \in F \mid h(x) \geq (1/3)a\}.$$

Since h is continuous, then $h^{-1}((-\infty, 1(1/3)a])$ and $h^{-1}([(1/3)a, \infty))$ are closed and so (since F ic closed) sets A and B are closed. Of course A and B are disjoint. Therefore, by Urysohn's Lemma, there is a continuous real-valued function g on X for which $|g| \leq (1/3)a$ on X, $g(A) = -(1/3)a$, and $g(B) = (1/3)a$. Since $|h| < a$ on F, then for $x \in A$, $h(x) \leq -(1/3)a$ and so $|h - g| \leq a - (1/3)a = (2/3)a$ on A; for $x \in B$, $h(x) \geq (1/3)a$ and so $|h - g| \leq a - (1/3)a = (2/3)a$ on B;

The Tietze Extension Theorem (continued 3)

The Tietze Extension Theorem.

Let $(X, \mathcal{T})$ be a normal topological space, F a closed subset of X, and f a continuous real-valued function on F that takes values in the closed, bounded interval $[a, b]$. Then f has a continuous extension to all of X that also takes values in $[a, b]$.

Proof (continued). since by Urysohn's Lemma, $g(x)$ is between $-(1/3)a$ and $(1/3)a$ (that is, $|g(x)| \leq (1/3)a$), then for all $x \notin A \cup B$ and $x \in F$ we have $|h(x) - g(x)| \leq (2/3)a$. So $|h - g| \leq (2/3)a$ on F. So function g satisfies the claim. With $a = 1$, choose such a g denoted $g - 1$ with $|g_1| \leq 2/3$ on X and $f - g_1| \leq 2/3$ on F. now iterate the above process with $h = f - g_1$ and $a = 2/3$ to find a continuous $g_2 : X \to \mathbb{R}$ for which $|g_2| \leq 2/3$ on X and $|f - (g_1 + g_2)| \leq (2/3)^2$ on F. We can then inductively construct the desired sequence $\{g_n\}$ which satisfies properties (1) and (2). The result now follows. $\square$

The Urysohn Metrization Theorem

The Urysohn Metrization Theorem.
Let $(X, \mathcal{T})$ be a second countable topological space. Then $(X, \mathcal{T})$ is metrizable if and only if it is normal.

Proof. If $(X, \mathcal{T})$ is metrizable then the result is a metric space. By Proposition 11.7, every metric space is normal.

Now let $(X, \mathcal{T})$ be a second countable and normal topological space. Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a countable base (of distinct sets) for topology $(X, \mathcal{T})$. Let $A \subseteq \mathbb{N}\times\mathbb{N}$ be defined as

$$A = \{(n,m) \in \mathbb{N}\times\mathbb{N} \mid \overline{\mathcal{U}_n} \subseteq \mathcal{U}_m\}.$$

For each $(n,m) \in A$ we see that $\overline{\mathcal{U}_n}$ and $X \sim \mathcal{U}_m$ are disjoint closed sets. Since $(X, \mathcal{T})$ is normal, Urysohn's Lemma there is a continuous real-valued function $f_{n,m} : X \to [0, 1]$ for which $f_{n,m} = 0$ on $\overline{\mathcal{U}_n}$ and $f_{n,m} = 1$ on $X \sim \mathcal{U}_m$.

The Urysohn Metrization Theorem (continued 1)

Proof (continued). For $x, y \in X$, define the (alleged) metric

$$\rho(x,y) = \sum_{(n,m)\in A} \frac{1}{2^{n+m}}|f_{n,m}(x) - f_{n,m}(y)|.$$

Notice that $|f_{n,m}(x) - f_{n,m}(y)| \leq 1$ for all $x, y \in X$. For $n \neq m$, we cannot have both (n,m) and (m,n) in A (or else $\overline{\mathcal{U}_n} \subseteq \mathcal{U}_m$ and $\overline{\mathcal{U}_m} \subseteq \mathcal{U}_n$, in which case $\mathcal{U}_n = \mathcal{U}_m$ which contradicts the fact that the sets in $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ are distinct). So the elements of set A include AT MOST the following:

$$
\begin{array}{lllll}
(1,1), & (1,2), & (1,3), & (1,4), & \cdots \\
(2,2), & (2,3), & (2,4), & \cdots \\
(3,3), & (3,4), & \cdots \\
(4,4), & \cdots \\
& & & & \ddots
\end{array}
$$

The Urysohn Metrization Theorem (continued 2)

Proof (continued). So for all $x, y \in X$ we have

$$
\begin{aligned}
\rho(x,y) &= \sum_{(n,m)\in A} \frac{1}{2^{n+m}}|f_{n,m}(x) - f_{n,m}(y)| \\
&\leq \sum_{(n,m)\in A} \frac{1}{2^{n+m}} \\
&\leq \sum_{m=1}^{n} \frac{1}{2^{1+m}} + \sum_{m=2}^{n} \frac{1}{2^{2+m}} + \sum_{m=3}^{n} \frac{1}{2^{3+m}} + \cdots \\
&= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots \\
&= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{2}\,\frac{1}{1 - 1/4} = \frac{2}{3}.
\end{aligned}
$$

So the series determining $\rho(x,y)$ converges.

The Urysohn Metrization Theorem (continued 3)

Proof (continued). Now to show that ρ is in fact a metric. Of course, $\rho(x,y) = \rho(y,x)$. Also, $\rho(x,y) \geq 0$ and $\rho(x,x) = 0$.

Claim 1. We claim $\rho(x,y) = 0$ implies $x = y$. We show the contrapositive. Suppose $x \neq y$. Since $(X,\mathcal{T})$ is normal (and hence, by definition, Tychonoff) then $\{x\}$ and $\{y\}$ are closed sets by Proposition 11.6. Since $(X, \mathcal{T})$ is normal, there is open $\mathcal{O}_x$ containing x and not containing y. So there is some base set $\mathcal{U}_m$ with $x \in \mathcal{U}_m$ and $\mathcal{U}_m \subseteq \mathcal{O}_x$. By Proposition 11.8, since $(X, \mathcal{T})$ is normal, there is open $\mathcal{O} \in \mathcal{T}$ such that $\{x\} \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}_m$. So there is a base set $\mathcal{U}_n$ with $x \in \mathcal{U}_n$ and $\mathcal{U}_n \subseteq \mathcal{O}$. Then $\overline{\mathcal{U}_n} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}_m$. So $f_{n,m}(x) = 0$ and, since $y \in X \sim \mathcal{U}_m$, $f_{n,m}(y) = 1$. Therefore, $(n, m) \in A$ and $|f_{n,m}(x) - f_{n,m}(y)| = 1$, so $\rho(x,y) \neq 0$, and the claim holds.

The Urysohn Metrization Theorem (continued 4)

Proof (continued). Claim 2. For all $x, y, z \in X$, we claim
$\rho(x, z) \le \rho(x, y) + \rho(y, z)$. For any $(n, m) \in A$ we have

$$
\begin{aligned}
|f_{n,m}(x) - f_{,m}(z)n, m(z)| &= |f_{n,m}(x) - f_{n,m}(y) + f_{n,m}(y) - f_{n,m}(z)| \\
&\le |f_{n,m}(x) - f_{n,m}(y)| + |f_{n,m}(y) - f_{n,m}(z)|
\end{aligned}
$$

by the Triangle Inequality on $\mathbb{R}$,

$$
\begin{aligned}
\rho(x, z) &= \sum_{(n,m)\in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(z)| \\
&\le \sum_{(n,m)\in A} \frac{1}{2^{n+m}} \left(|f_{n,m}(x) - f_{n,m}(y)| + |f_{n,m}(y) - f_{n,m}(z)| \right) \\
&= \sum_{(n,m)\in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)| \\
&\quad + \sum_{(n,m)\in A} \frac{1}{2^{n+m}} |f_{n,m}(y) - f_{n,m}(z)| = \rho(x, y) + \rho(y, z).
\end{aligned}
$$

The Urysohn Metrization Theorem (continued 5)

Proof (continued). So the Triangle Inequality holds and Claim 2 holds.

Therefore ρ is a metric. We now need to show that topology $\mathcal{T}$ on X is the same as the topology on X induced by metric ρ. To do so, we need to show that for each $x \in X$:

(i) If $\mathcal{U}_n$ contains x, then there is an $\varepsilon > 0$ for which
$B_\rho(x, \varepsilon) \subseteq \mathcal{U}_n$.

(ii) For each $\varepsilon > 0$, there is a $\mathcal{U}_n$ that contains x and
$\mathcal{U}_n \subseteq B_\rho(x, \varepsilon)$.

It then follows that a set is open in one topology if and only if it is open in the other topology. These two properties are verified in Problem 12.7. □