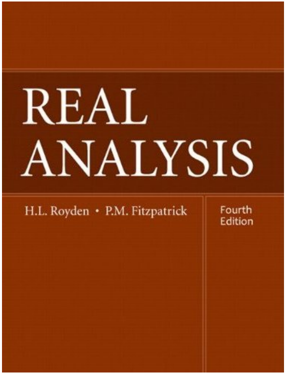
Real Analysis

Chapter 12. Topological Spaces: Three Fundamental Theorems

12.1. Urysohn's Lemma and the Tietze Extension Theorem—Proofs of Theorems



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$$\Lambda = \{m/2^n \mid m, n \in \mathbb{N}, 1 \leq m \leq 2^n - 1\}.$$

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Proof. This is $\mathcal{O}_\lambda$ for all $\lambda \in \Lambda_1 = \{1/2\}$. Again, by Proposition 11.8, with closed F and neighborhood $\mathcal{U} = \mathcal{O}_{1/2}$ of F there is open $\mathcal{O}_{1/4}$ with $F \subset \mathcal{O}_{1/4} \subset \bar{)}_{1/4} \subset \mathcal{O}_{1/2}$. With closed $\overline{\mathcal{O}}_{1/2}$ and neighborhood $\mathcal{U}$ of $\overline{\mathcal{O}}_{1/2}$ there is by Proposition 11.8 open $\mathcal{O}_{3/4}$ with $\overline{\mathcal{O}}_{1/2} \subset \mathcal{O}_{3/4} \subset \overline{\mathcal{O}}_{3/4} \subset \mathcal{U}$. So we have

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So the normally ascending collection $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda_1}$ is extended to normally ascending collection $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda_2}$. We then proceed inductively to define for each $n \in \mathbb{N}$, the normally ascending collection of open sets $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda_n}$.

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Urysohn's Lemma. Let A and B be disjoint closed subsets of a normal topological space $(X, \mathcal{T})$. Then for any closed bounded interval of real numbers $[a, b]$, there is a continuous real-valued function f defined on X that takes values in $[a, b]$, while $f = a$ on A and $f = b$ on B.

Proof. By Lemma 12.2, with $F = A$ and $\mathcal{U} = X \setminus B$, we can choose a dense subset Λ of (a, b) and a normally ascending collection of open subsets of X, $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$, for which $A \subset \mathcal{O}_\lambda \subset X \setminus B$ for all $\lambda \in \Lambda$.

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The Tietze Extension Theorem

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Let $(X, \mathcal{T})$ be a normal topological space, F a closed subset of X, and f a continuous real-valued function on F that takes values in the closed, bounded interval $[a, b]$. Then f has a continuous extension to all of X that also takes values in $[a, b]$.

Proof. Since $[a, b]$ and $[-2, 2]$ are homeomorphic (consider $f : [a, b] \to [-2, 2]$ defined as $f(x) = 4(x - a)/(b - a) - 2)$, we assume without loss of generality that $[a, b] = [-2, 2]$.

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We construct a sequence $\{g_n\}$ of continuous real-valued functions on X with the following properties:

(1) For each $n \in \mathbb{N}$, $|g_n(x)| \leq (2/3)^n$ on X, and

(2) for each $n \in \mathbb{N}$, $|f - (g_1 + g_2 + \cdots + g_n)| \leq (2/3)^n$ on F.

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Proof (continued). So define $g(x) = \lim_{n\to\infty} s_n(x)$ for each $x \in X$. Since each g_n is continuous on X then, of course, each s_n is continuous. By property (1), $\{s_n\}$ converges to g uniformly on X (since $\sum_{n=1}^{\infty}(2/3)^n = 2$). Therefore g is continuous on X. By property (2), f is also the (uniform) pointwise limit of $\{s_n\}$ on F, so $f = g$ on F, Notice that for each $x \in X$

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The Tietze Extension Theorem (continued 2)

Proof (continued). We claim that for any $a > 0$ and continuous function $h : F \to \mathbb{R}$ for which $|h| \leq a$ on F, there is a continuous function $g : X \to \mathbb{R}$ such that

$$|g| \leq (2/3)a \text{ on } X \text{ and } |j - g| \leq (2/3)a \text{ on } F.$$

We justify this claim by defining

$$A = \{x \in F \mid h(x) \leq 1(1/3)a\} \text{ and } B = x \in F \mid h(x) \geq (1/3)a\}.$$

Since h is continuous, then $h^{-1}((-\infty, 1(1/3)a])$ and $h^{-1}([(1/3)a, \infty))$ are closed and so (since F ic closed) sets A and B are closed.

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The Urysohn Metrization Theorem

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Let $(X, \mathcal{T})$ be a second countable topological space. Then $(X, \mathcal{T})$ is metrizable if and only if it is normal.

Proof. If $(X, \mathcal{T})$ is metrizable then the result is a metric space. By Proposition 11.7, every metric space is normal.

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Proof. If $(X, \mathcal{T})$ is metrizable then the result is a metric space. By Proposition 11.7, every metric space is normal.

Now let $(X, \mathcal{T})$ be a second countable and normal topological space. Let $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ be a countable base (of distinct sets) for topology $(X, \mathcal{T})$. Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be defined as

$$A = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid \overline{\mathcal{U}}_n \subseteq \mathcal{U}_m\}.$$

The Urysohn Metrization Theorem

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Let $(X, \mathcal{T})$ be a second countable topological space. Then $(X, \mathcal{T})$ is metrizable if and only if it is normal.

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For each $(n, m) \in A$ we see that $\overline{\mathcal{U}}_n$ and $X \sim \mathcal{U}_m$ are disjoint closed sets. Since $(X, \mathcal{T})$ is normal, Urysohn's Lemma there is a continuous real-valued function $f_{n,m} : X \to [0, 1]$ for which $f_{n,m} = 0$ on $\overline{\mathcal{U}}_n$ and $f_{n,m} = 1$ on $X \sim \mathcal{U}_m$.

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The Urysohn Metrization Theorem (continued 1)

Proof (continued). For $x, y \in X$, define the (alleged) metric

$$\rho(x, y) = \sum_{(n,m)\in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)|.$$

Notice that $|f_{n,m}(x) - f_{n,m}(y)| \leq 1$ for all $x, y \in X$. For $n \neq m$, we cannot have both (n, m) and (m, n) in A (or else $\overline{\mathcal{U}}_n \subseteq \mathcal{U}_m$ and $\overline{\mathcal{U}}_m \subseteq \mathcal{U}_n$, in which case $\mathcal{U}_n = \mathcal{U}_m$ which contradicts the fact that the sets in $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ are distinct).

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$$\begin{array}{ccccc} (1,1), & (1,2), & (1,3), & (1,4), & \cdots \\ & (2,2), & (2,3), & (2,4), & \cdots \\ & & (3,3), & (3,4), & \cdots \\ & & & (4,4), & \cdots \\ & & & & \ddots \end{array}$$

The Urysohn Metrization Theorem (continued 1)

Proof (continued). For $x, y \in X$, define the (alleged) metric

$$\rho(x,y) = \sum_{(n,m)\in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)|.$$

Notice that $|f_{n,m}(x) - f_{n,m}(y)| \le 1$ for all $x, y \in X$. For $n \ne m$, we cannot have both (n,m) and (m,n) in A (or else $\overline{\mathcal{U}}_n \subseteq \mathcal{U}_m$ and $\overline{\mathcal{U}}_m \subseteq \mathcal{U}_n$, in which case $\mathcal{U}_n = \mathcal{U}_m$ which contradicts the fact that the sets in $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ are distinct). So the elements of set A include AT MOST the following:

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The Urysohn Metrization Theorem (continued 2)

Proof (continued). So for all $x, y \in X$ we have

$$\begin{aligned} \rho(x,y) &= \sum_{(n,m)\in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)| \\ &\leq \sum_{(n,m)\in A} \frac{1}{2^{n+m}} \\ &\leq \sum_{m=1}^{n} \frac{1}{2^{1+m}} + \sum_{m=2}^{n} \frac{1}{2^{2+m}} + \sum_{m=3}^{n} \frac{1}{2^{3+m}} + \cdots \\ &= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{2} \frac{1}{1 - 1/4} = \frac{2}{3}. \end{aligned}$$

So the series determining $\rho(x, y)$ converges.

The Urysohn Metrization Theorem (continued 3)

Proof (continued). Now to show that ρ is in fact a metric. Of course, $\rho(x, y) = \rho(y, x)$. Also, $\rho(x, y) \geq 0$ and $\rho(x, x) = 0$.

Claim 1. We claim $\rho(x, y) = 0$ implies $x = y$. We show the contrapositive. Suppose $x \neq y$. Since$(X, \mathcal{T})$ is normal (and hence, by definition, Tychonoff) then $\{x\}$ and $\{y\}$ are closed sets by Proposition 11.6. Since $(X, \mathcal{T})$ is normal, there is open $\mathcal{O}_x$ containing x and not containing y. So there is some base set $\mathcal{U}_m$ with $x \in \mathcal{U}_m$ and $\mathcal{U}_m \subseteq \mathcal{O}_x$.

The Urysohn Metrization Theorem (continued 3)

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The Urysohn Metrization Theorem (continued 3)

Proof (continued). Now to show that ρ is in fact a metric. Of course, $\rho(x,y) = \rho(y,x)$. Also, $\rho(x,y) \geq 0$ and $\rho(x,x) = 0$.

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The Urysohn Metrization Theorem (continued 4)

Proof (continued). Claim 2. For all $x, y, z \in X$, we claim $\rho(x,z) \le \rho(x,y) + \rho(y,z)$. For any $(n,m) \in A$ we have

$$\begin{aligned} |f_{n,m}(x) - f)n,m(z)| &= |f_{n,m}(x) - f_{n,m}(y) + f_{n,m}(y) - f_{n,m}(z)| \\ &\le |f_{n,m}(x) - f_{n,m}(y)| + |f_{n,m}(y) - f_{n,m}(z)| \\ &\quad \text{by the Triangle Inequality on } \mathbb{R}, \end{aligned}$$

$$\begin{aligned} \rho(x,z) &= \sum_{(n,m)\in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(z)| \\ &\le \sum_{(n,m)\in A} \frac{1}{2^{n+m}} \left(|f_{n,m}(x) - f_{n,m}(y)| + |f_{n,m}(y) - f_{n,m}(z)\right) \\ &= \sum_{(n,m)\in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)| \\ &\quad + \sum_{(n,m)\in A} \frac{1}{2^{n+m}} |f_{n,m}(y) - f_{n,m}(z)| = \rho(x,y) + \rho(y,z). \end{aligned}$$

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The Urysohn Metrization Theorem (continued 5)

Proof (continued). So the Triangle Inequality holds and Claim 2 holds.

Therefore ρ is a metric. We now need to show that topology $\mathcal{T}$ on X is the same as the topology on X induced by metric ρ. To do so, we need to show that for each $x \in X$:

(i) If $\mathcal{U}_n$ contains x, then there is an $\varepsilon > 0$ for which $B_\rho(x, \varepsilon) \subseteq \mathcal{U}_n$.

(ii) For each $\varepsilon > 0$, there is a $\mathcal{U}_n$ that contains x and $\mathcal{U}_n \subseteq B_\rho(x, \varepsilon)$.

It then follows that a set is open in one topology if and only if it is open in the other topology. These two properties are verified in Problem 12.7. □

The Urysohn Metrization Theorem (continued 5)

Proof (continued). So the Triangle Inequality holds and Claim 2 holds.

Therefore ρ is a metric. We now need to show that topology $\mathcal{T}$ on X is the same as the topology on X induced by metric ρ. To do so, we need to show that for each $x \in X$:

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