Chapter 12. Topological Spaces: Three Fundamental Theorems
12.1. Urysohn’s Lemma and the Tietze Extension Theorem—Proofs of Theorems
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Lemma 12.2

**Lemma 12.2.** Let \((X, \mathcal{T})\) be a normal topological space, \(F\) a closed subset of \(X\), and \(\mathcal{U}\) a neighborhood of \(F\). Then for any open, bounded interval \((a, b)\), there is a dense subset \(\Lambda\) of \((a, b)\) and a normally ascending collection of open subsets of \(X\), \(\{O_\lambda\}_{\lambda \in \Lambda}\), for which

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F \subseteq O_\lambda \subseteq \overline{O}_\lambda \subseteq \mathcal{U} \text{ for all } \lambda \in \Lambda.
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**Proof.** Without loss of generality, we take \((a, b) = (0, 1)\) (otherwise we continuously map \((a, b)\) to \((0, 1)\) with \(f(x) = (x - a)/(b - a)\) and then apply the result we now prove).
Lemma 12.2. Let \((X, \mathcal{T})\) be a normal topological space, \(F\) a closed subset of \(X\), and \(U\) a neighborhood of \(F\). Then for any open, bounded interval \((a, b)\), there is a dense subset \(\Lambda\) of \((a, b)\) and a normally ascending collection of open subsets of \(X\), \(\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}\), for which

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\[
\Lambda = \{m/2^n \mid m, n \in \mathbb{N}, 1 \leq m \leq 2^n - 1\}.
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Let

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\Lambda_n = \{m/2^n \mid m \in \mathbb{N}, 1 \leq m \leq 2^n - 1\}.
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By Proposition 11.8, there is open $O_{1/2}$ for which $F \subseteq O_{1/2} \subseteq \overline{O_{1/2}} \subseteq U$. 

\[ \text{Real Analysis} \]
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Lemma 12.2 (continued)

Proof. This is $O_\lambda$ for all $\lambda \in \Lambda_1 = \{1/2\}$. Again, by Proposition 11.8, with closed $F$ and neighborhood $U = O_{1/2}$ of $F$ there is open $O_{1/4}$ with $F \subset O_{1/4} \subset \overline{O}_{1/4} \subset O_{1/2}$. With closed $\overline{O}_{1/2}$ and neighborhood $U$ of $\overline{O}_{1/2}$ there is by Proposition 11.8 open $O_{3/4}$ with $\overline{O}_{1/2} \subset O_{3/4} \subset \overline{O}_{3/4} \subset U$. So we have

$$F \subset O_{1/4} \subset \overline{O}_{1/4} \subset O_{1/2} \subset \overline{O}_{1/2} \subset O_{3/4} \subset \overline{O}_{3/4} \subset U.$$
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So the normally ascending collection $\{O_\lambda\}_{\lambda \in \Lambda_1}$ is extended to normally ascending collection $\{O_\lambda\}_{\lambda \in \Lambda_2}$. We then proceed inductively to define for each $n \in \mathbb{N}$, the normally ascending collection of open sets $\{O_\lambda\}_{\lambda \in \Lambda_n}$. 
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Urysohn’s Lemma

**Urysohn’s Lemma.** Let $A$ and $B$ be disjoint closed subsets of a normal topological space $(X, T)$. Then for any closed bounded interval of real numbers $[a, b]$, there is a continuous real-valued function $f$ defined on $X$ that takes values in $[a, b]$, while $f = a$ on $A$ and $f = b$ on $B$.

**Proof.** By Lemma 12.2, with $F = A$ and $U = X \setminus B$, we can choose a dense subset $\Lambda$ of $(a, b)$ and a normally ascending collection of open subsets of $X$, $\{O_\lambda\}_{\lambda \in \Lambda}$, for which $A \subset O_\lambda \subset X \setminus B$ for all $\lambda \in \Lambda$. 
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Urysohn’s Lemma. Let $A$ and $B$ be disjoint closed subsets of a normal topological space $(X, T)$. Then for any closed bounded interval of real numbers $[a, b]$, there is a continuous real-valued function $f$ defined on $X$ that takes values in $[a, b]$, while $f = a$ on $A$ and $f = b$ on $B$.

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The Tietze Extension Theorem

Let $(X, T)$ be a normal topological space, $F$ a closed subset of $X$, and $f$ a continuous real-valued function on $F$ that takes values in the closed, bounded interval $[a, b]$. Then $f$ has a continuous extension to all of $X$ that also takes values in $[a, b]$.

**Proof.** Since $[a, b]$ and $[-2, 2]$ are homeomorphic (consider $f : [a, b] \to [-2, 2]$ defined as $f(x) = 4(x - a)/(b - a) - 2$), we assume without loss of generality that $[a, b] = [-2, 2]$. 
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We construct a sequence \(\{g_n\}\) of continuous real-valued functions on \(X\) with the following properties:

1. For each \(n \in \mathbb{N}\), \(|g_n(x)| \leq (2/3)^n\) on \(X\), and
2. for each \(n \in \mathbb{N}\), \(|f - (g_1 + g_2 + \cdots + g_n)| \leq (2/3)^n\) on \(F\).

With this sequence constructed, define for each \(n \in \mathbb{N}\), the real-valued function \(s_n\) on \(X\) by \(s_n(x) = \sum_{k=1}^{n} g_k(x)\) for \(x \in X\).
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**Proof (continued).** So define \( g(x) = \lim_{n \to \infty} s_n(x) \) for each \( x \in X \). Since each \( g_n \) is continuous on \( X \) then, of course, each \( s_n \) is continuous. By property (1), \( \{s_n\} \) converges to \( g \) uniformly on \( X \) (since \( \sum_{n=1}^{\infty} (2/3)^n = 2 \)). Therefore \( g \) is continuous on \( X \). By property (2), \( f \) is also the (uniform) pointwise limit of \( \{s_n\} \) on \( F \), so \( f = g \) on \( F \). Notice that for each \( x \in X \)

\[
g(x) = \left| \sum_{k=1}^{\infty} g_n(x) \right| \leq \sum_{k=1}^{\infty} |g_n(x)| \leq \sum_{k=1}^{\infty} (2/3)^n = 2.
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So \( g(x) \in [-2, 2] \) for all \( x \in X \). Therefore \( g \) is the desired continuous extension of \( f \) to \( X \).
Proof (continued). So define $g(x) = \lim_{n \to \infty} s_n(x)$ for each $x \in X$. Since each $g_n$ is continuous on $X$ then, of course, each $s_n$ is continuous. By property (1), $\{s_n\}$ converges to $g$ uniformly on $X$ (since $\sum_{n=1}^{\infty} (2/3)^n = 2$). Therefore $g$ is continuous on $X$. By property (2), $f$ is also the (uniform) pointwise limit of $\{s_n\}$ on $F$, so $f = g$ on $F$, Notice that for each $x \in X$

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So $g(x) \in [-2, 2]$ for all $x \in X$. Therefore $g$ is the desired continuous extension of $f$ to $X$. We now construct the sequence $\{g_n\}$. 

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$$g(x) = \left| \sum_{k=1}^{\infty} g_n(x) \right| \leq \sum_{k=1}^{\infty} |g_n(x)| \leq \sum_{k=1}^{\infty} (2/3)^n = 2.$$ 

So $g(x) \in [-2, 2]$ for all $x \in X$. Therefore $g$ is the desired continuous extension of $f$ to $X$. We now construct the sequence \{g_n\}. 

The Tietze Extension Theorem (continued 1)
The Tietze Extension Theorem (continued 2)

Proof (continued). We claim that for any $a > 0$ and continuous function $h : F \to \mathbb{R}$ for which $|h| \leq a$ on $F$, there is a continuous function $g : X \to \mathbb{R}$ such that

$$|g| \leq \frac{2}{3}a \text{ on } X \text{ and } |j - g| \leq \frac{2}{3}a \text{ on } F.$$  

We justify this claim by defining

$$A = \{x \in F \mid h(x) \leq 1(1/3)a\} \text{ and } B = \{x \in F \mid h(x) \geq (1/3)a\}.$$

Since $h$ is continuous, then $h^{-1}((-\infty, 1(1/3)a])$ and $h^{-1}([(1/3)a, \infty))$ are closed and so (since $F$ is closed) sets $A$ and $B$ are closed.
The Tietze Extension Theorem (continued 2)

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The Tietze Extension Theorem (continued 2)

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The Tietze Extension Theorem.
Let \((X, T)\) be a normal topological space, \(F\) a closed subset of \(X\), and \(f\) a continuous real-valued function on \(F\) that takes values in the closed, bounded interval \([a, b]\). Then \(f\) has a continuous extension to all of \(X\) that also takes values in \([a, b]\).

Proof (continued). since by Urysohn’s Lemma, \(g(x)\) is between \(-\left(\frac{1}{3}\right)a\) and \(\left(\frac{1}{3}\right)a\) (that is, \(|g(x)| \leq \left(\frac{1}{3}\right)a\)), then for all \(x \notin A \cup B\) and \(x \in F\) we have \(|h(x) - g(x)| \leq \left(\frac{2}{3}\right)a\). So \(|h - g| \leq \left(\frac{2}{3}\right)a\) on \(F\). So function \(g\) satisfies the claim. With \(a = 1\), choose such a \(g\) denoted \(g - 1\) with \(|g_1| \leq \frac{2}{3}\) on \(X\) and \(|f - g_1| \leq \frac{2}{3}\) on \(F\). now iterate the above process with \(h = f - g_1\) and \(a = \frac{2}{3}\) to find a continuous \(g_2 : X \to \mathbb{R}\) for which \(|g_2| \leq \frac{2}{3}\) on \(X\) and \(|f - (g_1 + g_2)| \leq \left(\frac{2}{3}\right)^2\) on \(F\). We can then inductively construct the desired sequence \(\{g_n\}\) which satisfies properties (1) and (2). The result now follows.
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Let \((X, T)\) be a normal topological space, \(F\) a closed subset of \(X\), and \(f\) a continuous real-valued function on \(F\) that takes values in the closed, bounded interval \([a, b]\). Then \(f\) has a continuous extension to all of \(X\) that also takes values in \([a, b]\).

Proof (continued). Since by Urysohn’s Lemma, \(g(x)\) is between \(-\frac{1}{3}a\) and \(\frac{1}{3}a\) (that is, \(|g(x)| \leq \frac{1}{3}a\)), then for all \(x \notin A \cup B\) and \(x \in F\) we have \(|h(x) - g(x)| \leq \frac{2}{3}a\). So \(|h - g| \leq \frac{2}{3}a\) on \(F\). So function \(g\) satisfies the claim. With \(a = 1\), choose such a \(g\) denoted \(g - 1\) with \(|g_1| \leq \frac{2}{3}\) on \(X\) and \(|f - g_1| \leq \frac{2}{3}\) on \(F\). Now iterate the above process with \(h = f - g_1\) and \(a = \frac{2}{3}\) to find a continuous \(g_2 : X \to \mathbb{R}\) for which \(|g_2| \leq \frac{2}{3}\) on \(X\) and \(|f - (g_1 + g_2)| \leq (\frac{2}{3})^2\) on \(F\). We can then inductively construct the desired sequence \(\{g_n\}\) which satisfies properties (1) and (2). The result now follows.
The Urysohn Metrization Theorem

Let \((X, T)\) be a second countable topological space. Then \((X, T)\) is metrizable if and only if it is normal.

**Proof.** If \((X, T)\) is metrizable then the result is a metric space. By Proposition 11.7, every metric space is normal.
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Now let \((X, \mathcal{T})\) be a second countable and normal topological space. Let \(\{\mathcal{U}_n\}_{n \in \mathbb{N}}\) be a countable base (of distinct sets) for topology \((X, \mathcal{T})\). Let \(A \subseteq \mathbb{N} \times \mathbb{N}\) be defined as

\[
A = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid \overline{\mathcal{U}}_n \subseteq \mathcal{U}_m\}.
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For each \( (n, m) \in A \) we see that \( \overline{U_n} \) and \( X \sim U_m \) are disjoint closed sets. Since \( (X, \mathcal{T}) \) is normal, Urysohn’s Lemma there is a continuous real-valued function \( f_{n,m} : X \to [0, 1] \) for which \( f_{n,m} = 0 \) on \( \overline{U_n} \) and \( f_{n,m} = 1 \) on \( X \sim U_m \).
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The Urysohn Metrization Theorem (continued 1)

Proof (continued). For \( x, y \in X \), define the (alleged) metric

\[
\rho(x, y) = \sum_{(n,m) \in A} \frac{1}{2^{n+m}}|f_{n,m}(x) - f_{n,m}(y)|.
\]

Notice that \( |f_{n,m}(x) - f_{n,m}(y)| \leq 1 \) for all \( x, y \in X \). For \( n \neq m \), we cannot have both \( (n, m) \) and \( (m, n) \) in \( A \) (or else \( \overline{U}_n \subseteq \overline{U}_m \) and \( \overline{U}_m \subseteq \overline{U}_n \), in which case \( U_n = U_m \) which contradicts the fact that the sets in \( \{U_n\}_{n \in \mathbb{N}} \) are distinct).
Proof (continued). For \( x, y \in X \), define the (alleged) metric

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\[
(1, 1), \ (1, 2), \ (1, 3), \ (1, 4), \ \cdots \\
(2, 2), \ (2, 3), \ (2, 4), \ \cdots \\
(3, 3), \ (3, 4), \ \cdots \\
(4, 4), \ \cdots \\
\cdots
\]
The Urysohn Metrization Theorem (continued 1)

Proof (continued). For \( x, y \in X \), define the (alleged) metric

\[
\rho(x, y) = \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)|.
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\[
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(3,3), \ (3,4), \ \ldots \\
(4,4), \ \ldots \\
\ldots
\]
Proof (continued). So for all $x, y \in X$ we have

$$\rho(x, y) = \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)|$$

$$\leq \sum_{(n,m) \in A} \frac{1}{2^{n+m}}$$

$$\leq \sum_{m=1}^{n} \frac{1}{2^{1+m}} + \sum_{m=2}^{n} \frac{1}{2^{2+m}} + \sum_{m=3}^{n} \frac{1}{2^{3+m}} + \cdots$$

$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{2} \cdot \frac{1}{1 - 1/4} = \frac{2}{3}.$$ 

So the series determining $\rho(x, y)$ converges.
Proof (continued). Now to show that $\rho$ is in fact a metric. Of course, $\rho(x, y) = \rho(y, x)$. Also, $\rho(x, y) \geq 0$ and $\rho(x, x) = 0$.

Claim 1. We claim $\rho(x, y) = 0$ implies $x = y$. We show the contrapositive. Suppose $x \neq y$. Since $(X, \mathcal{T})$ is normal (and hence, by definition, Tychonoff) then $\{x\}$ and $\{y\}$ are closed sets by Proposition 11.6. Since $(X, \mathcal{T})$ is normal, there is open $\mathcal{O}_x$ containing $x$ and not containing $y$. So there is some base set $\mathcal{U}_m$ with $x \in \mathcal{U}_m$ and $\mathcal{U}_m \subseteq \mathcal{O}_x$. 


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Proof (continued). Now to show that $\rho$ is in fact a metric. Of course, $\rho(x, y) = \rho(y, x)$. Also, $\rho(x, y) \geq 0$ and $\rho(x, x) = 0$.

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The Urysohn Metrization Theorem (continued 3)

Proof (continued). Now to show that $\rho$ is in fact a metric. Of course, $\rho(x, y) = \rho(y, x)$. Also, $\rho(x, y) \geq 0$ and $\rho(x, x) = 0$.

Claim 1. We claim $\rho(x, y) = 0$ implies $x = y$. We show the contrapositive. Suppose $x \neq y$. Since $(X, T)$ is normal (and hence, by definition, Tychonoff) then $\{x\}$ and $\{y\}$ are closed sets by Proposition 11.6. Since $(X, T)$ is normal, there is open $O_x$ containing $x$ and not containing $y$. So there is some base set $U_m$ with $x \in U_m$ and $U_m \subseteq O_x$. By Proposition 11.8, since $(X, T)$ is normal, there is open $O \in T$ such that $\{x\} \subseteq O \subseteq \overline{O} \subseteq U_m$. So there is a base set $U_n$ with $x \in U_n$ and $U_n \subseteq O$. Then $\overline{U}_n \subseteq \overline{O} \subseteq U_m$. So $f_{n,m}(x) = 0$ and, since $y \in X \sim U_m$, $f_{n,m}(y) = 1$. Therefore, $(n, m) \in A$ and $|f_{n,m}(x) - f_{n,m}(y)| = 1$, so $\rho(x, y) \neq 0$, and the claim holds.
The Urysohn Metrization Theorem (continued 4)

**Proof (continued). Claim 2.** For all \( x, y, z \in X \), we claim \( \rho(x, z) \leq \rho(x, y) + \rho(y, z) \). For any \((n, m) \in A\) we have

\[
|f_{n,m}(x) - f_{n,m}(z)| = |f_{n,m}(x) - f_{n,m}(y) + f_{n,m}(y) - f_{n,m}(z)| \\
\leq |f_{n,m}(x) - f_{n,m}(y)| + |f_{n,m}(y) - f_{n,m}(z)|
\]

by the Triangle Inequality on \( \mathbb{R} \),

\[
\rho(x, z) = \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(z)| \\
\leq \sum_{(n,m) \in A} \frac{1}{2^{n+m}} (|f_{n,m}(x) - f_{n,m}(y)| + |f_{n,m}(y) - f_{n,m}(z)|) \\
= \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)| \\
+ \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(y) - f_{n,m}(z)| = \rho(x, y) + \rho(y, z). 
\]
The Urysohn Metrization Theorem (continued 4)

**Proof (continued). Claim 2.** For all \( x, y, z \in X \), we claim \( \rho(x, z) \leq \rho(x, y) + \rho(y, z) \). For any \((n, m) \in A\) we have

\[
|f_{n,m}(x) - f_{n,m}(z)| = |f_{n,m}(x) - f_{n,m}(y) + f_{n,m}(y) - f_{n,m}(z)| \\
\leq |f_{n,m}(x) - f_{n,m}(y)| + |f_{n,m}(y) - f_{n,m}(z)|
\]

by the Triangle Inequality on \( \mathbb{R} \),

\[
\rho(x, z) = \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(z)|
\]

\[
\leq \sum_{(n,m) \in A} \frac{1}{2^{n+m}} (|f_{n,m}(x) - f_{n,m}(y)| + |f_{n,m}(y) - f_{n,m}(z)|)
\]

\[
= \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)| + \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(y) - f_{n,m}(z)| = \rho(x, y) + \rho(y, z).
\]
Proof (continued). So the Triangle Inequality holds and Claim 2 holds.

Therefore \( \rho \) is a metric. We now need to show that topology \( \mathcal{T} \) on \( X \) is the same as the topology on \( X \) induced by metric \( \rho \). To do so, we need to show that for each \( x \in X \):

(i) If \( U_n \) contains \( x \), then there is an \( \varepsilon > 0 \) for which \( B_\rho(x, \varepsilon) \subseteq U_n \).

(ii) For each \( \varepsilon > 0 \), there is a \( U_n \) that contains \( x \) and \( U_n \subseteq B_\rho(x, \varepsilon) \).

It then follows that a set is open in one topology if and only if it is open in the other topology. These two properties are verified in Problem 12.7.
Proof (continued). So the Triangle Inequality holds and Claim 2 holds. Therefore $\rho$ is a metric. We now need to show that topology $T$ on $X$ is the same as the topology on $X$ induced by metric $\rho$. To do so, we need to show that for each $x \in X$:

(i) If $U_n$ contains $x$, then there is an $\varepsilon > 0$ for which $B_\rho(x, \varepsilon) \subseteq U_n$.

(ii) For each $\varepsilon > 0$, there is a $U_n$ that contains $x$ and $U_n \subseteq B_\rho(x, \varepsilon)$.

It then follows that a set is open in one topology if and only if it is open in the other topology. These two properties are verified in Problem 12.7.