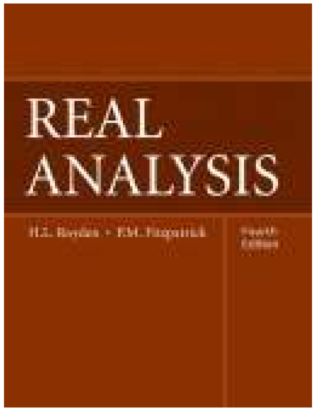
Real Analysis

Chapter 12. Topological Spaces: Three Fundamental Theorems
12.2. The Tychonoff Product Theorem—Proofs of Theorems



Lemma 12.5

Lemma 12.5. Let $\mathcal{A}$ be a collection of subsets of a set X that possesses the finite intersection property. Then there is a collection $\mathcal{B}$ of subsets of X which contains $\mathcal{A}$, has the finite intersection property, and is maximal with respect to this property; that is, no collection of subsets of X that properly contains $\mathcal{B}$ possesses the finite intersection property.

Proof. Consider the family $\mathcal{F}$ of all collections of subsets of X which contain $\mathcal{A}$ and possessing the finite intersection property. Partially order $\mathcal{F}$ by set inclusion. Every linearly ordered subfamily $\mathcal{F}_0$ of $\mathcal{F}$ has an upper bound consisting of the union of all the sets in $\mathcal{F}_0$. Then by Zorn's Lemma set $\mathcal{F}$ has a maximal member. This maximal element is the set $\mathcal{B}$ in the lemma. □

Lemma 12.6

Lemma 12.6. Let $\mathcal{B}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then each intersection of a finite property. Then each intersection of a finite number of sets in $\mathcal{B}$ is again in $\mathcal{B}$, and each subset of X that has nonempty intersection with each set in $\mathcal{B}$ is itself in $\mathcal{B}$.

Proof. Let $\mathcal{B}'$ be the collection of all sets that are finite intersections of sets in $\mathcal{B}$. Then $\mathcal{B}'$ is a collection of sets with the finite intersection property and $\mathcal{B} \subset \mathcal{B}'$. Since $\mathcal{B}$ is maximal with respect to this property, then $\mathcal{B}' = \mathcal{B}$. Now suppose that C is a subset of X that has nonempty intersection with each member of $\mathcal{B}$. Since $\mathcal{B}$ contains each finite intersection of sets in $\mathcal{B}$ (by the $\mathcal{B}'$ argument above) then $\mathcal{B} \cup \{C\}$ has the finite intersection property (a finite collection of sets in $\mathcal{B}$ has nonempty intersection and this intersection itself is in $\mathcal{B}$, so C has nonempty intersection with the finite intersection of elements in $\mathcal{B}$). Since $\mathcal{B}$ is maximal, then $\mathcal{B} = \mathcal{B} \cup \{C\}$ and so $C \in \mathcal{B}$ as claimed. □

The Tychonoff Product Theorem

The Tychonoff Product Theorem.
Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a collection of compact topological spaces indexed by a set Λ. Then the Cartesian product $\prod_{\lambda\in\Lambda} X_\lambda$, with the product topology, also is compact.

Proof. Let $\mathcal{F}$ be a collection of closed subsets of $X = \prod_{\lambda\in\Lambda} X_\lambda$ possessing the finite intersection property. The result will follow by Proposition 11.14 if we can show that the intersection of all elements of $\mathcal{F}$ is nonempty. By Lemma 12.5, there is a collection $\mathcal{B}$ of (not necessarily closed) subsets of X that contains $\mathcal{F}$ and is maximal with respect to the finite intersection property. Fix $\lambda \in \Lambda$. Define $\mathcal{B}_\lambda = \{\pi_\lambda(B) \mid B \in \mathcal{B}\}$ (that is, $\mathcal{B}_\lambda$ consists of the projection of all $B \in \mathcal{B}$ onto X_λ). Then $\mathcal{B}_\lambda$ is a collection of subsets of the set X_λ that has the finite intersection property (for if the finite intersection property were not satisfied in the λ-th component then it would not be satisfied over all). So the closure of the members of $\mathcal{B}_\lambda$ also have the finite intersection property.

The Tychonoff Product Theorem (continued 1)

Proof (continued). Since X_λ is compact, by Proposition 11.14 there is a point $x_\lambda \in X_\lambda$ for which $x_\lambda \in \cap_{B\in\mathcal{B}}\overline{\pi_\lambda(B)}$. Define x to be the point in X whose λ-th coordinate is x_λ. We claim that

$$x \in \cap_{F\in\mathcal{F}} F. \qquad (7)$$

Indeed, the point x has the property that for each index λ, x_λ is a point of closure of $\pi_\lambda(B)$ for every $B \in \mathcal{B}$.

We define an open set of the form $\mathcal{O} = \prod_{\lambda\in\Lambda} \mathcal{O}_\lambda$ where each $\mathcal{O}_\lambda = X_\lambda$ except for one value of λ, say λ_0, where $\mathcal{O}_{\lambda_0}$ is an open set in X_{λ_0}, as a *subbasic set*. The finite intersection of subbasic sets is a *basic set* (for which all but finitely many $\mathcal{O}_\lambda = X_\lambda$ and the remaining $\mathcal{O}_\lambda$ are open in the respective space). Thus every subbasic neighborhood $\mathcal{N}_x$ of x has a nonempty intersection with every $B \in \mathcal{B}$. By Lemma 12.6, since $\mathcal{B}$ is maximal, every subbasic neighborhood of x is in $\mathcal{B}$. So every basic neighborhood of x (being a finite intersection of subbasic neighborhoods of x) is, again by Lemma 12.6, in $\mathcal{B}$.

The Tychonoff Product Theorem (continued 2)

The Tychonoff Product Theorem.
Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a collection of compact topological spaces indexed by a set Λ. Then the Cartesian product $\prod_{\lambda\in\Lambda} X_\lambda$, with the product topology, also is compact.

Proof (continued). But $\mathcal{B}$ (by choice) has the finite intersection property and contains the collection $\mathcal{F}$. Let F be a set in $\mathcal{F}$. Then F is in $\mathcal{B}$ and so every basic neighborhood of x (which is in $\mathcal{B}$ as explained above) has nonempty intersection with F (applying the finite intersection property to the basic neighborhood and set F). Hence x is a point of closure of the closed set F (since the basic open sets are a base for the product topology by definition of product topology). Hence, $x \in F$ and (7) holds. So the intersection of all elements of $\mathcal{F}$ is nonempty and by Proposition 11.4, X is compact. □