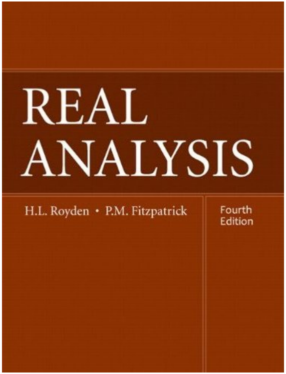
Real Analysis

Chapter 12. Topological Spaces: Three Fundamental Theorems

12.2. The Tychonoff Product Theorem—Proofs of Theorems



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Proof. Consider the family $\mathcal{F}$ of all collections of subsets of X which contain $\mathcal{A}$ and possessing the finite intersection property. Partially order $\mathcal{F}$ by set inclusion. Every linearly ordered subfamily $\mathcal{F}_0$ of $\mathcal{F}$ has an upper bound consisting of the union of all the sets in $\mathcal{F}_0$.

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Proof. Let $\mathcal{B}'$ be the collection of all sets that are finite intersections of sets in $\mathcal{B}$. Then $\mathcal{B}'$ is a collection of sets with the finite intersection property and $\mathcal{B} \subset \mathcal{B}'$. Since $\mathcal{B}$ is maximal with respect to this property, then $\mathcal{B}' = \mathcal{B}$.

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Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a collection of compact topological spaces indexed by a set Λ. Then the Cartesian product $\prod_{\lambda \in \Lambda} X_\lambda$, with the product topology, also is compact.

Proof. Let $\mathcal{F}$ be a collection of closed subsets of $X = \prod_{\lambda \in \Lambda} X_\lambda$ possessing the finite intersection property. The result will follow by Proposition 11.14 if we can show that the intersection of all elements of $\mathcal{F}$ is nonempty. By Lemma 12.5, there is a collection $\mathcal{B}$ of (not necessarily closed) subsets of X that contains $\mathcal{F}$ and is maximal with respect to the finite intersection property.

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The Tychonoff Product Theorem (continued 1)

Proof (continued). Since X_λ is compact, by Proposition 11.14 there is a point $x_\lambda \in X_\lambda$ for which $x_\lambda \in \cap_{B \in \mathcal{B}} \overline{\pi_\lambda(B)}$. Define x to be the point in X whose λ-th coordinate is x_λ. We claim that

$$x \in \cap_{F \in \mathcal{F}} F. \tag{7}$$

Indeed, the point x has the property that for each index λ, x_λ is a point of closure of $\pi_\lambda(B)$ for every $B \in \mathcal{B}$.

We define an open set of the form $\mathcal{O} = \prod_{\lambda \in \Lambda} \mathcal{O}_\lambda$ where each $\mathcal{O}_\lambda = X_\lambda$ except for one value of λ, say λ_0, where $\mathcal{O}_{\lambda_0}$ is an open set in X_{λ_0}, as a *subbasic set*. The finite intersection of subbasic sets is a *basic set* (for which all but finitely many $\mathcal{O}_\lambda = X_\lambda$ and the remaining $\mathcal{O}_\lambda$ are open in the respective space).

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