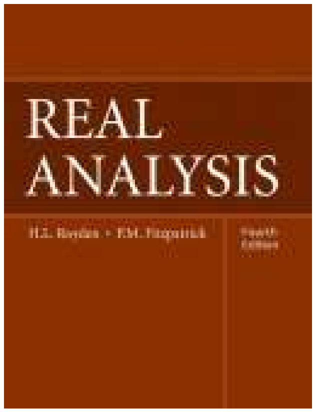
Real Analysis

Chapter 12. Topological Spaces: Three Fundamental Theorems

12.3. The Stone-Weierstrass Theorem—Proofs of Theorems



Lemma 12.7

Lemma 12.7. Let X be a compact Hausdorff space and $\mathcal{A}$ an algebra of continuous functions on X that separates points and contains the constant functions. Then for each closed subset F of X and point $x_0 \in X \sim F$, there is a neighborhood $\mathcal{U}$ of x_0 that is disjoint from F and has the following property: For each $\varepsilon > 0$ there is a function $h \in \mathcal{A}$ for which

$$h < \varepsilon \text{ on } \mathcal{U},\ h > 1-\varepsilon \text{ on } F, \text{ and } 0 \le h \le 1 \text{ on } X.$$

Proof. Since $\mathcal{A}$ separates points, for each $y \in F$ there is $f \in \mathcal{A}$ for which $f(x_0) \neq f(y)$. The function

$$g_y = \left(\frac{f - f(x_0)}{\|f - f(x_0)\|_{\max}} \right)^2$$

in in $\mathcal{A}$ since $\mathcal{A}$ is a linear space closed under products containing constant functions (by the definition of "algebra"). Also, $g(x_0) = 0$, $g(y) > 0$, and $0 \le g_y \le 1$ on X. Since g_y is continuous, there is a neighborhood $\mathcal{N}_y$ of y on which g_y only takes on positive values.

Lemma 12.7 (continued 1)

Proof (continued). However, F is a closed subset of the compact space X and so F itself is compact by Proposition 11.15. Since this holds for each $y \in F$, we can find a finite collection $\{y_1, y_2, \ldots, y_n\}$ of points in F, with corresponding g_{y_i}'s such that the $\mathcal{N}_{y_i}$'s cover G. Define the function $g \in \mathcal{A}$ by

$$g = \frac{1}{n} \sum_{i=1}^{n} g_{y_i}.$$

Then $g(x_0) = 0$, $g > 0$ on F, and $0 \le g \le 1$ on X. But a continuous function on a compact set takes on a minimum value (Corollary 11.21), so we may choose $c > 0$ for which $g \ge c$ on F. By possibly multiplying g by a positive number, we may suppose $c < 1$. On the other hand, g is continuous at x_0 where $g(x_0) = 0$, so there is a neighborhood $\mathcal{U}$ of x_0 for which

$$g < c/2 \text{ on } \mathcal{U},\ g \ge c \text{ on } F, \text{ and } 0 \le g \le 1 \text{ on } X.$$

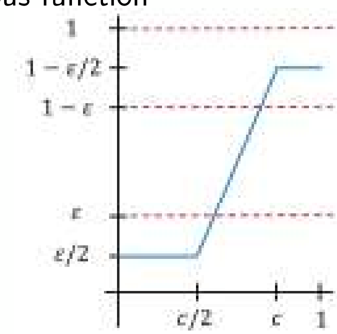
We claim that (10) holds for this choice of neighborhood $\mathcal{U}$.

Lemma 12.7 (continued 2)

Proof (continued). Let $\varepsilon > 0$. By the Weierstrass Approximation Theorem, there is a polynomial p such that

$$p < \varepsilon \text{ on } [0, c/2],\ p > 1 - \varepsilon \text{ on } [c, 1], \text{ and } 0 \le p \le 1 \text{ on } [0, 1]. \quad (14)$$

Consider the continuous function



and apply the Weierstrass Approximation Theorem for $\varepsilon/2 < 0$ to get the desired polynomial p.

Lemma 12.7 (continued 3)

Lemma 12.7. Let X be a compact Hausdorff space and $\mathcal{A}$ an algebra of continuous functions on X that separates points and contains the constant functions. Then for each closed subset F of X and point $x_0 \in X \sim F$, there is a neighborhood $\mathcal{U}$ of x_0 that is disjoint from F and has the following property: For each $\varepsilon > 0$ there is a function $h \in \mathcal{A}$ for which

$$h < \varepsilon \text{ on } \mathcal{U},\ h > 1-\varepsilon \text{ on } F, \text{ and } 0 \le h \le 1 \text{ on } X.$$

Proof (continued). Since p is a polynomial and g belongs to $\mathcal{A}$, the composition $h = p \circ g$ also belongs to $\mathcal{A}$ ($\mathcal{A}$ is closed under products and it a linear space). From (13) and (14) we have

- $g < c/2$ on $\mathcal{U}$ and $p < \varepsilon$ on $[0, c/2]$ implies $h = p \circ g < \varepsilon$ on $\mathcal{U}$,
- $g \ge c/2$ on F and $p > 1 - \varepsilon$ on $[c, 1]$ implies $h = p \circ g > 1 - \varepsilon$ on F, and
- $0 \le g \le 1$ on X and $0 \le p \le 1$ on $[0, 1]$ implies $h = p \circ g$ satisfies $0 \le h \le 1$ on X.

So (10) holds for h and $\mathcal{U}$. □

Lemma 12.8

Lemma 12.8. Let X be a compact Hausdorff space and $\mathcal{A}$ an algebra of continuous functions on X that separates points and contains the constant functions. Then for each pair of disjoint closed subsets A and B of X and $\varepsilon > 0$, there is a function h belonging to $\mathcal{A}$ for which

$$h < \varepsilon \text{ on } A,\ h > 1 - \varepsilon \text{ on } B, \text{ and } 0 \le h \le 1 \text{ on } X.$$

Proof. By Lemma 12.7 with $F = B$, we know that for each $x \in A$ there is a neighborhood $\mathcal{N}_x$ of x that is disjoint from B and has the property (10). However A is a closed subset of compact space X and so A is compact (Proposition 11.15), so there is a finite subset of points $\{x_1, x_2, \ldots, x_n\}$ in A with corresponding neighborhoods $\mathcal{N}_{x_1}, \mathcal{N}_{x_2}, \cdots, \mathcal{N}_{x_n}$ that covers A. Choose ε_0 for which $0 < \varepsilon_0 < \varepsilon$ and $(1 - \varepsilon_0/n)^n > 1 - \varepsilon$. For $1 \le i \le n$, since $\mathcal{N}_{x_i}$ has property (1) with $B = F$, we choose $h_i \in \mathcal{A}$ such that $h_i < \varepsilon_0/n$ on $\mathcal{N}_x$, $h_i > 1 - \varepsilon_0/n$ on B, and $0 \le j_i \le 1$ on X. Define $h = h_1 h_2 \cdots h_n$ on X. Then h belongs to $\mathcal{A}$ (since $\mathcal{A}$ is closed under products).

Lemma 12.8 (continued)

Lemma 12.8. Let X be a compact Hausdorff space and $\mathcal{A}$ an algebra of continuous functions on X that separates points and contains the constant functions. Then for each pair of disjoint closed subsets A and B of X and $\varepsilon > 0$, there is a function h belonging to $\mathcal{A}$ for which

$$h < \varepsilon \text{ on } A,\ h > 1 - \varepsilon \text{ on } B, \text{ and } 0 \le h \le 1 \text{ on } X.$$

Proof. Since for each i we have $0 \le h_i \le 1$ on X, then $0 \le j \le 1$ on X. Also, for each i we have $h_i > 1 - \varepsilon/n$ on B, so $h \ge (1 - \varepsilon/n)^n > 1 - \varepsilon$ on B. Finally, for each point $x \in A$ there is an index i for which $x \in \mathcal{N}_{x_i}$ (since the $\mathcal{N}_{x_i}$ form a cover). Thus $h_i(x) < \varepsilon_0/n < \varepsilon$ and since for the other indices j we have $0 \le h_j(x) \le 1$, it follows that $h(x) < \varepsilon_0/n < \varepsilon$. So function h has the desired properties. □

The Stone-Weierstrass Approximation Theorem

The Stone-Weierstrass Approximation Theorem.
Let X be a compact Hausdorff space. Suppose $\mathcal{A}$ is an algebra of continuous real-valued functions on X that separates points in X and contains the constant functions. Then $\mathcal{A}$ is dense in $C(X)$.

Proof. Let f belong to $C(X)$ and let $c = \|f\|_{\max}$. If we can arbitrarily closely uniformly approximate the function $(f + c)/\|f + c\|_{\max}$ by functions in $\mathcal{A}$, we can do the same for f (take the function approximating this, multiply if by the constant $\|f + c\|_{\max}$, and then subtract the constant c). Therefore, we may assume $0 \le f \le 1$ on X. Let $n \in \mathbb{N}$, $n > 1$. Consider the uniform partition $\{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$ of $[0, 1]$. Fix j with $1 \le j \le n$. Define

$$A_j = x \in X \mid f(x) \le (j-1)/n\} \text{ and } B_j = \{x \in X \mid f(x) \ge j/n\}.$$

Since f is continuous, both $A_j = f^{-1}((-\infty, (j-1)/n])$ and $B_j = f^{-1}([j/n, \infty))$ are closed subsets of X which are disjoint.

The Stone-Weierstrass Approximation Theorem (cont. 1)

Proof (continued). By Lemma 12.8, with $A = A_j$, $B = B_j$, and $\varepsilon = 1/n$, there is a function $g_j \in \mathcal{A}$ for which

$$g_j(x) < 1/n \text{ if } f(x) \leq (j-1)/n \text{ (i.e., } x \in A_j),$$

$$g_j(x) > 1 - 1/n \text{ if } f(x) \geq j/n \text{ (i.e., } x \in B_j), \text{ and } 0 \leq g_j \leq 1 \text{ on } X. \quad (16)$$

Define $g = (1/n)\sum_{j=1}^n g_j$. Then $g \in \mathcal{A}$. We claim that

$$\|f - g\|_{\max} < 3/n. \quad (17)$$

With this established, given $\varepsilon > 0$ we just select n such that $3/n < \varepsilon$ and then $\{f - g\|_{\max} < \varepsilon$ as desired. To verify (17) we first show that

$$\text{if } 1 \leq k \leq n \text{ and } f(x) \leq k/n \text{ then } g(x) \leq k/n + 1/n. \quad (18)$$

The Stone-Weierstrass Approximation Theorem (cont. 2)

Proof (continued). Indeed, for $j = k+1, k+2, \ldots, n$, with the assumption that $f(x) \leq k/n$, we have that $f(x) \leq k/n \leq (j-1)/n$. Therefore $g_j(x) \leq 1/n$ by (16). Thus

$$\frac{1}{n}\sum_{j=k+1}^{n} g_j \leq \frac{1}{n}\left(\frac{n-k}{n}\right) \leq \frac{1}{n}. \quad (18')$$

Consequently, since each $g_j(x) \leq 1$ by (16), for all k

$$\begin{aligned} g(x) &= \frac{1}{n}\sum_{j=1}^{n} g_j(x) = \frac{1}{n}\sum_{j=1}^{k} g_j(x) + \frac{1}{n}\sum_{j=k+1}^{n} g_j(x) \\ &\leq \frac{1}{n}\sum_{j=1}^{k} g_j(x) + \frac{1}{n} \text{ by } (18') \\ &\leq \frac{k}{n} + \frac{1}{n} \text{ since } g_j(x) \leq 1. \end{aligned}$$

Thus (18) holds.

The Stone-Weierstrass Approximation Theorem (cont. 3)

Proof (continued). We now show that

$$\text{if } 1 \leq k \leq n \text{ and } (k-1)/n \leq f(x) \text{ then } (k-1)/n = 1/n \leq g(x). \quad (19)$$

Indeed, for $j = 1, 2, \ldots, k-1$ with the assumption that $(k-1)/n \leq f(z)$, we have that $j/n \leq (k-1)/n \leq f(x)$. Therefore $1 - a/n < g_j(x)$ by (16). Thus

$$\frac{1}{n}\sum_{j=1}^{k-1} g_i(x) > \frac{k-1}{n}\left(1 - \frac{1}{n}\right) = \frac{k-1}{n} - \frac{k-1}{n^2}. \quad (18'')$$

Now $(k-1)/n^2 \leq n/n^2 = 1/n$, so $-(k-1)/n^2 \geq -1/n$ and

$$\frac{1}{n}\sum_{j=1}^{k-1} g_j(x) \geq \frac{k-1}{n} - \frac{k-1}{n^2} > \frac{k-1}{n} - \frac{1}{n}.$$

The Stone-Weierstrass Approximation Theorem (cont. 4)

Proof (continued). Consequently, since each $g_j(x) \geq 0$ by (16), for all k

$$g(x) = \frac{1}{n}\sum_{j=1}^{n} g_j(x) \geq \frac{1}{n}\sum_{j=1}^{k-1} g_j(x) \geq \frac{k-1}{n} - \frac{1}{n}. \quad (19)$$

For $x \in X$, choose k with $1 \leq k \leq n$, such that $(k-1)/n \leq f(x) \leq k/n$ (since $0 \leq f(x) \leq 1$ for all $x \in X$ without loss of generality, as stated above, there is such k). From (18) and (19),

$$f(x) \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \text{ and } g(x) \in \left[\frac{k-1}{n} - \frac{1}{n}, \frac{k}{n} + \frac{1}{n}\right],$$

so $|f(x) - g(x)| \leq 2/n < 3/n$ (consider the extremes $f(x) = (k-1)/n$ and $g(x) = k/n + 1/n$ AND $f(x) = k/n$ and $g(x) = (k-1)/n - 1/n$). So (17) holds and the result follows. □

Borsuk's Theorem

Borsuk's Theorem.
Let X be a compact Hausdorff topological space. Then $C(X)$ is separable if and only if X is metrizable.

Proof. First, assume X is metrizable with metric ρ that induces the topology on X. Then X, being a compact metric space, is separable by Proposition 9.24. Choose a countable dense subset $\{x_n\}$ of X. For each $n \in \mathbb{N}$, define $f_n(x) = \rho(x, x_n)$ for all $x \in X$. Since ρ induces the topology, f_n is continuous (Think: If x varies a little then $\rho(x, x_n)$ varies a little and so $f_n(x)$ varies a little). Let $u, v \in X$, $u \neq v$. Since $\{x_n\}$ is dense in X, there are x_u, x_v in $\{x_n\}$ such that $\rho(x_u, u) < \rho(u, v)/2$ and $\rho(x_v, v) < \rho(u, v)/2$. Then $f_{x_u}(u) < \rho(u, x)/2$ and $f_{x_u}(v) > \rho(u, v)/2$, so $f_{x_u}(u) \neq f_{x_u}(v)$ and $\{f_n\}$ separates points in X. Define $f_0 \equiv 1$ on X. Let $\mathcal{A}$ be the collection of polynomials with real coefficients in a finite number of the f_k (that is, take polynomials in several variables and evaluate them at some of the f_k).

Borsuk's Theorem (continued 1)

Proof (continued). Then $\mathcal{A}$ is an algebra that contains the constant functions and it separates points in X since it contains the f_k. By the Stone-Weierstrass Theorem, $\mathcal{A}$ is dense in $C(X)$. But the collection of functions in $\mathcal{A}$ that are polynomials with rational coefficients is a countable set that is dense in $\mathcal{A}$. Therefore $C(X)$ is separable.

Conversely, suppose $C(X)$ is separable. Let $\{g_n\}$ be a countable dense subset of $C(X)$. For each $n \in \mathbb{N}$ define $\mathcal{O}_n = \{x \in X \mid g_n(x) > 1/2\}$. Then $\{\mathcal{O}_n\}$ is a countable collection of open sets. We now show X is second countable. Let $x \in \mathcal{O}$ where $\mathcal{O}$ is open. Since X is normal (because X is compact and Hausdorff, normality follows from Theorem 11.18), by Proposition 11.8 (since $\{x\}$ is a closed set by Proposition 11.6) there is an open set $\mathcal{U}$ for which $x \in \mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{O}$.

Borsuk's Theorem (continued 2)

Borsuk's Theorem.
Let X be a compact Hausdorff topological space. Then $C(X)$ is separable if and only if X is metrizable.

Proof. By Urysohn's Lemma there is a g in $C(X)$ such that $g(x) = 1$ on $\mathcal{U} \subset \overline{\mathcal{U}}$ is dense in $C(X)$, there is $n \in \mathbb{N}$ such that $|g - g_n| < 1/2$ on X. So $g_n(x) > 1/2$ on $\mathcal{U}$ (since $g(x) = 1$ on $\mathcal{U}$). Hence $x \in \mathcal{U} \subset \mathcal{O}_n \subset \mathcal{O}$. So $\{\mathcal{O}_n\}$ is a countable base for the topological space; that is, X is second countable. So by the Urysohn Metrization Theorem, X is metrizable. □