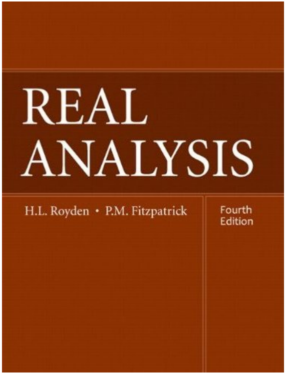
Real Analysis

Chapter 12. Topological Spaces: Three Fundamental Theorems

12.3. The Stone-Weierstrass Theorem—Proofs of Theorems



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$$h < \varepsilon \text{ on } \mathcal{U},\ h > 1-\varepsilon \text{ on } F,\ \text{and } 0 \leq h \leq 1 \text{ on } X.$$

Proof. Since $\mathcal{A}$ separates points, for each $y \in F$ there is $f \in \mathcal{A}$ for which $f(x_0) \neq f(y)$. The function

$$g_y = \left(\frac{f - f(x_0)}{\|f - f(x_0)\|_{\max}} \right)^2$$

in in $\mathcal{A}$ since $\mathcal{A}$ is a linear space closed under products containing constant functions (by the definition of "algebra").

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Lemma 12.7 (continued 1)

Proof (continued). However, F is a closed subset of the compact space X and so F itself is compact by Proposition 11.15. Since this holds for each $y \in F$, we can find a finite collection $\{y_1, y_2, \ldots, y_n\}$ of points in F, with corresponding g_{y_i}'s such that the $\mathcal{N}_{y_i}$'s cover G. Define the function $g \in \mathcal{A}$ by

$$g = \frac{1}{n} \sum_{i=1}^{n} g_{y_i}.$$

Then $g(x_0) = 0$, $g > 0$ on F, and $0 \leq g \leq 1$ on X. But a continuous function on a compact set takes on a minimum value (Corollary 11.21), so we may choose $c > 0$ for which $g \geq c$ on F. By possibly multiplying g by a positive number, we may suppose $c < 1$.

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$$g < c/2 \text{ on } \mathcal{U},\ g \geq c \text{ on } F, \text{ and } 0 \leq g \leq 1 \text{ on } X.$$

We claim that (10) holds for this choice of neighborhood $\mathcal{U}$.

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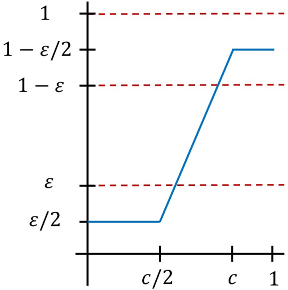

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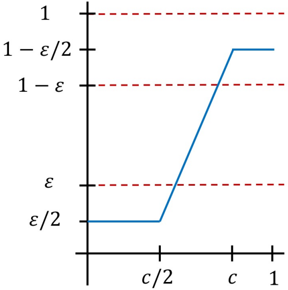

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Proof (continued). Since p is a polynomial and g belongs to $\mathcal{A}$, the composition $h = p \circ g$ also belongs to $\mathcal{A}$ ($\mathcal{A}$ is closed under products and it a linear space). From (13) and (14) we have

- $g < c/2$ on $\mathcal{U}$ and $p < \varepsilon$ on $[0, c/2]$ implies $h = p \circ g < \varepsilon$ on $\mathcal{U}$,
- $g \geq c/2$ on F and $p > 1 - \varepsilon$ on $[c, 1]$ implies $h = p \circ g > 1 - \varepsilon$ on F, and
- $0 \leq g \leq 1$ on X and $0 \leq p \leq 1$ on $[0, 1]$ implies $h = p \circ g$ satisfies $0 \leq h \leq 1$ on X.

So (10) holds for h and $\mathcal{U}$. □

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Let X be a compact Hausdorff space. Suppose $\mathcal{A}$ is an algebra of continuous real-valued functions on X that separates points in X and contains the constant functions. Then $\mathcal{A}$ is dense in $C(X)$.

Proof. Let f belong to $C(X)$ and let $c = \|f\|_{\max}$. If we can arbitrarily closely uniformly approximate the function $(f + c)/\|f + c\|_{\max}$ by functions in $\mathcal{A}$, we can do the same for f (take the function approximating this, multiply if by the constant $\|f + c\|_{\max}$, and then subtract the constant c). Therefore, we may assume $0 \leq f \leq 1$ on X.

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$$A_j = x \in X \mid f(x) \le (j-1)/n\} \text{ and } B_j = \{x \in X \mid f(x) \ge j/n\}.$$

Since f is continuous, both $A_j = f^{-1}((-\infty, (j-1)/n])$ and $B_j = f^{-1}([j/n, \infty))$ are closed subsets of X which are disjoint.

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Proof (continued). By Lemma 12.8, with $A = A_j$, $B = B_j$, and $\varepsilon = 1/n$, there is a function $g_j \in \mathcal{A}$ for which

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Define $g = (1/n) \sum_{j=1}^{n} g_j$. Then $g \in \mathcal{A}$. We claim that

$$\|f - g\|_{\max} < 3/n. \quad (17)$$

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With this established, given $\varepsilon > 0$ we just select n such that $3/n < \varepsilon$ and then $\{f - g\|_{\max} < \varepsilon$ as desired. To verify (17) we first show that

$$\text{if } 1 \leq k \leq n \text{ and } f(x) \leq k/n \text{ then } g(x) \leq k/n + 1/n. \quad (18)$$

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Proof (continued). Indeed, for $j = k+1, k+2, \ldots, n$, with the assumption that $f(x) \leq k/n$, we have that $f(x) \leq k/n \leq (j-1)/n$. Therefore $g_j(x) \leq 1/n$ by (16). Thus

$$\frac{1}{n} \sum_{j=k+1}^{n} g_j \leq \frac{1}{n}\left(\frac{n-k}{n}\right) \leq \frac{1}{n}. \quad (18')$$

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Consequently, since each $g_j(x) \leq 1$ by (16), for all k

$$\begin{aligned} g(x) &= \frac{1}{n}\sum_{j=1}^{n} g_j(x) = \frac{1}{n}\sum_{j=1}^{k} g_j(x) + \frac{1}{n}\sum_{j=k+1}^{n} g_j(x) \\ &\leq \frac{1}{n}\sum_{j=1}^{k} g_j(x) + \frac{1}{n} \text{ by } (18') \\ &\leq \frac{k}{n} + \frac{1}{n} \text{ since } g_j(x) \leq 1. \end{aligned}$$

Thus (18) holds.

The Stone-Weierstrass Approximation Theorem (cont. 2)

Proof (continued). Indeed, for $j = k+1, k+2, \ldots, n$, with the assumption that $f(x) \le k/n$, we have that $f(x) \le k/n \le (j-1)/n$. Therefore $g_j(x) \le 1/n$ by (16). Thus

$$\frac{1}{n}\sum_{j=k+1}^{n} g_j \le \frac{1}{n}\left(\frac{n-k}{n}\right) \le \frac{1}{n}. \qquad (18')$$

Consequently, since each $g_j(x) \le 1$ by (16), for all k

$$\begin{aligned} g(x) &= \frac{1}{n}\sum_{j=1}^{n} g_j(x) = \frac{1}{n}\sum_{j=1}^{k} g_j(x) + \frac{1}{n}\sum_{j=k+1}^{n} g_j(x) \\ &\le \frac{1}{n}\sum_{j=1}^{k} g_j(x) + \frac{1}{n} \text{ by } (18') \\ &\le \frac{k}{n} + \frac{1}{n} \text{ since } g_j(x) \le 1. \end{aligned}$$

Thus (18) holds.

The Stone-Weierstrass Approximation Theorem (cont. 3)

Proof (continued). We now show that

$$\text{if } 1 \leq k \leq n \text{ and } (k-1)/n \leq f(x) \text{ then } (k-1)/n = 1/n \leq g(x). \quad (19)$$

Indeed, for $j = 1, 2, \ldots, k-1$ with the assumption that $(k-1)/n \leq f(z)$, we have that $j/n \leq (k-1)/n \leq f(x)$. Therefore $1 - a/n < g_j(x)$ by (16). Thus

$$\frac{1}{n}\sum_{j=1}^{k-1} g_i(x) > \frac{k-1}{n}\left(1 - \frac{1}{n}\right) = \frac{k-1}{n} - \frac{k-1}{n^2}. \quad (18'')$$

Now $(k-1)/n^2 \leq n/n^2 = 1/n$, so $-(k-1)/n^2 \geq -1/n$ and

$$\frac{1}{n}\sum_{j=1}^{k-1} g_j(x) \geq \frac{k-1}{n} - \frac{k-1}{n^2} > \frac{k-1}{n} - \frac{1}{n}.$$

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The Stone-Weierstrass Approximation Theorem (cont. 4)

Proof (continued). Consequently, since each $g_j(x) \geq 0$ by (16), for all k

$$g(x) = \frac{1}{n}\sum_{j=1}^{n} g_j(x) \geq \frac{1}{n}\sum_{j=1}^{k-1} g_j(x) \geq \frac{k-1}{n} - \frac{1}{n}. \qquad (19)$$

For $x \in X$, choose k with $1 \leq k \leq n$, such that $(k-1)/n \leq f(x) \leq k/n$ (since $0 \leq f(x) \leq 1$ for all $x \in X$ without loss of generality, as stated above, there is such k). From (18) and (19),

$$f(x) \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \text{ and } g(x) \in \left[\frac{k-1}{n} - \frac{1}{n}, \frac{k}{n} + \frac{1}{n}\right],$$

so $|f(x) - g(x)| \leq 2/n < 3/n$ (consider the extremes $f(x) = (k-1)/n$ and $g(x) = k/n + 1/n$ AND $f(x) = k/n$ and $g(x) = (k-1)/n - 1/n$). So (17) holds and the result follows. □

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Borsuk's Theorem

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Let X be a compact Hausdorff topological space. Then $C(X)$ is separable if and only if X is metrizable.

Proof. First, assume X is metrizable with metric ρ that induces the topology on X. Then X, being a compact metric space, is separable by Proposition 9.24. Choose a countable dense subset $\{x_n\}$ of X.

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Borsuk's Theorem (continued 1)

Proof (continued). Then $\mathcal{A}$ is an algebra that contains the constant functions and it separates points in X since it contains the f_k. By the Stone-Weierstrass Theorem, $\mathcal{A}$ is dense in $C(X)$. But the collection of functions in $\mathcal{A}$ that are polynomials with rational coefficients is a countable set that is dense in $\mathcal{A}$. Therefore $C(X)$ is separable.

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Conversely, suppose $C(X)$ is separable. Let $\{g_n\}$ be a countable dense subset of $C(X)$. For each $n \in \mathbb{N}$ define $\mathcal{O}_n = \{x \in X \mid g_n(x) > 1/2\}$. Then $\{\mathcal{O}_n\}$ is a countable collection of open sets.

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Borsuk's Theorem (continued 2)

Borsuk's Theorem.
Let X be a compact Hausdorff topological space. Then $C(X)$ is separable if and only if X is metrizable.

Proof. By Urysohn's Lemma there is a g in $C(X)$ such that $g(x) = 1$ on $\mathcal{U} \subset \overline{\mathcal{U}}$ is dense in $C(X)$, there is $n \in \mathbb{N}$ such that $|g - g_n| < 1/2$ on X. So $g_n(x) > 1/2$ on $\mathcal{U}$ (since $g(x) = 1$ on $\mathcal{U}$). Hence $x \in \mathcal{U} \subset \mathcal{O}_n \subset \mathcal{O}$. So $\{\mathcal{O}_n\}$ is a countable base for the topological space; that is, X is second countable. So by the Urysohn Metrization Theorem, X is metrizable. □

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