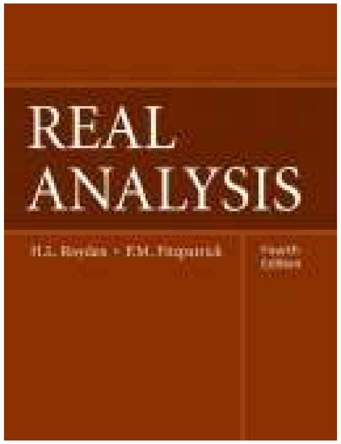
# Real Analysis

**Chapter 13. Continuous Linear Operators on Hilbert Between Banach Spaces**

13.2. Linear Operators—Proofs of Theorems








## Theorem 13.1

**Theorem 13.1** A linear operator between normed linear spaces is continuous if and only if it is bounded.

**Proof.** Let $X$ and $Y$ be normed linear spaces and $T : X \to Y$ be linear.

Suppose $T$ is bounded. Then by definition $\|T(u)\| \le \|T\|\|u\|$ for all $u \in X$ and so $\|T(u) - T(v)\| = \|T(u-v)\| \le \|T\|\|u - v\|$ for all $u, v \in X$. So if $u_0 \in X$ then for any $\varepsilon > 0$, with $\delta = \varepsilon/\|T\|$ we have for all $u \in X$ where $\|u_0 - u\| < \delta$ than

$$\|T(u_0) - T(u)\| \le \|T\|\|u_0 - u\| < \|T\|\delta = \|T\|\varepsilon/\|T\| = \varepsilon.$$

So $T$ is continuous at $u_0 \in X$ and since $u_0$ is arbitrary then $T$ is continuous on $X$.

Now suppose $T : X \to Y$ is continuous. Since $T$ is linear then $T(0) = T(0+0) = T(0) + T(0)$ and so $T(0) = 0$. Let $\varepsilon = 1$. Since $T$ is continuous at $u = 0$ then there is $\delta > 0$ such that if $\|u - 0\| < \delta$ then $\|T(u) - T(0)\| = \|T(u)\| < 1$.





## Theorem 13.1 (continued)

**Theorem 13.1** A linear operator between normed linear spaces is continuous if and only if it is bounded.

**Proof (continued).** For any $u \in X$ where $u \neq 0$, let $\lambda = \delta/(2\|u\|)$. So $\|\lambda u\| = |\lambda|\|u\| = \lambda\|u\| = \delta/2 < \delta$. Thus $\|T(\lambda u)\| < 1$. Since $\|T(\lambda u)\| = \|\lambda T(u)\| = \lambda\|T(u)\|\delta/(2\|u\|) < 1$ then $\|T(u)\| = 2\|u\|/\delta - (2/\delta)\|u\|$ and so $\|T(u)\| \le M\|u\|$ where $M = 2/\delta$ for all $u \in X$. Therefore, $T$ is bounded. □





## Proposition 13.2

**Proposition 13.2.** Let $X$ and $Y$ be normed linear spaces. Then the collection of bounded linear operators from $X$ to $Y$, $\mathcal{L}(X,Y)$, is itself a normed linear space.

**Proof.** Let $T, S \in \mathcal{L}(X,Y)$. Then

$$\begin{aligned} \|(T+S)(u)\| &= \|T(u) + S(u)\| \\ &\le \|T(u)\| + \|S(u)\| \text{ by the Triangle Inequality on } \mathbb{R} \\ &\le \|T\|\|u\| + \|S\|\|u\| \text{ by the definition of operator norm} \\ &= (\|T\| + \|S\|)\|u\| \end{aligned}$$

and so $T + S$ is bounded by $\|T\| + \|S\|$. So $\mathcal{L}(X,Y)$ is closed under addition. For $\alpha \in \mathbb{R}$, $\|\alpha T(u)\| = |\alpha|\|T(u)\|$ by definition of $\|\cdot\|$ and so by Exercise 13.11, $\|\alpha T\| = \sup\{\|\alpha T(u)\| \mid u \in X, \|u\| \le 1\}$

$$= \sup\{|\alpha|\|T(u)\| \mid u \in X, \|u\| \le 1\} = |\alpha| \sup\{\|T(u)\| \mid u \in X, \|u\| \le 1\}$$
$$= \alpha\|T\|.$$

## Proposition 13.2 (continued)

**Proposition 13.2.** Let $X$ and $Y$ be normed linear spaces. Then the collection of bounded linear operators from $X$ to $Y$, $\mathcal{L}(X,Y)$, is itself a normed linear space.

**Proof (continued).** Finally, $\|T\| = 0$ means $\|T(u)\| \le 0\|u\| = 0$ and so $T(u) = 0$ for all $u \in X$. If $T(u) = 0$ for all $u \in X$ then $\|T\| = \sup\{\|T(u)\| \mid u \in X, \|u\| \le 1\} = 0$. Hence $\|T\| = 0$ if and only if $T = 0$. Therefore, $\|\cdot\|$ is a norm on $\mathcal{L}(X,Y)$ an d$\mathcal{L}(X,Y)$ is a normed linear space. □





## Theorem 13.3

**Theorem 13.3.** Let $X$ and $Y$ be normed linear spaces. If $Y$ is a Banach space, then so is $\mathcal{L}(X,Y)$.

**Proof.** Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(X,Y)$. Let $u \in X$. For all $m,n \in \mathbb{N}$, $\|T_n(u) - T_m(u)\| = \|(T_n - T_m)(u)\| \le \|T_n - T_m\|\|u\|$. So $\{T_n(u)\}$ is a Cauchy sequence in $Y$. Since $Y$ is complete, then $T_n(u) \to T(u)$ for some $T(u) \in Y$. So the resulting $T : X \to Y$ and $T$ is the "pointwise" limit of $T_n$. We need to prove that $T \in \mathcal{L}(X,Y)$ and $T_n \to T$ with respect to the norm in $\mathcal{L}(X,Y)$. Let $u_1, u_2 \in X$. Then

$$T(u_1) + T(u_2) = \lim_{n\to\infty} T_n(u_1) + \lim_{n\to\infty} T_n(u_2)$$

$$= \lim_{n\to\infty} (T_n(u_1) + T_n(u_2)) = \lim_{n\to\infty} T_n(u_1 + u_2) = T(u_1 + u_2).$$

Similarly, for $u \in X$ and $\lambda \in \mathbb{R}$,

$$T(\lambda u) = \lim_{n\to\infty} T(\lambda u) = \lim_{n\to\infty} \lambda T_n(u) = \lambda \lim_{n\to\infty} T_n(u) = \lambda T(u).$$





## Theorem 13.3 (continued)

**Proof (continued).** Combining these two results gives $T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$ and so $T$ is linear. Let $\varepsilon > 0$. Since $\{T_n\}$ is Cauchy in $\mathcal{L}(X,Y)$, choose $N \in \mathbb{N}$ such that for $n \ge N$ and $k \ge 1$ we have $\|T_n - T_{n+k}\| < \varepsilon/2$. Then for all $u \in X$,

$$\|T_n(u) - T_{n+k}(u)\| = \|(T_n - T_{n+k})(u)\| \le \|T_n - T_{n+k}\|\|u\| < \varepsilon\|u\|/2.$$

Fix $n \ge N$ and $u \in X$. Since $\lim_{k\to\infty} T_{n+k}(u) = T(u)$ and the norm is continuous then for $u \in X$,

$$\lim_{k\to\infty} \|T_n(u) - T_{n+k}(u)\| = \left\|\lim_{n\to\infty} (T_n(u) - T_{n+k}(u))\right\|$$

$$= \|T_n(u) - T(u)\| \le \varepsilon\|u\|/2,$$

and so $T_N - T$ is bounded (by $\varepsilon/2$). Since $T_N$ is bounded then $\|T\| = \|T_n - T + T\| \le \|T_n - T\| + \|T\| \le \|T\| + \varepsilon/2$ and so $T$ is bounded. Therefore $T \in \mathcal{L}(X,Y)$. Since $\varepsilon > 0$ is arbitrary and for this given $\varepsilon > 0$ we have $\|T_n - T\| < \varepsilon$ for $n \ge N$, then $\|T_n\| \to T$ is $\mathcal{L}(X,Y)$. □