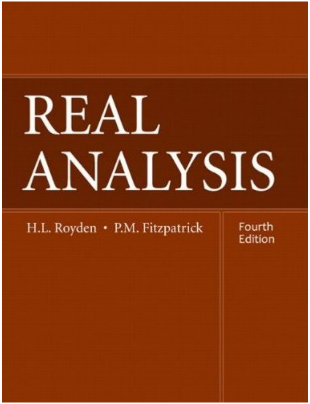
Real Analysis

Chapter 13. Continuous Linear Operators on Hilbert Between Banach Spaces

13.2. Linear Operators—Proofs of Theorems



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Theorem 13.1

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Suppose T is bounded. Then by definition $\|T(u)\| \leq \|T\|\|u\|$ for all $u \in X$ and so $\|T(u) - T(v)\| = \|T(u - v)\| \leq \|T\|\|u - v\|$ for all $u, v \in X$. So if $u_0 \in X$ then for any $\varepsilon > 0$, with $\delta = \varepsilon/\|T\|$ we have for all $u \in X$ where $\|u_0 - u\| < \delta$ than

$$\|T(u_0) - T(u)\| \leq \|T\|\|u_0 - u\| < \|T\|\delta = \|T\|\varepsilon/\|T\| = \varepsilon.$$

So T is continuous at $u_0 \in X$ and since u_0 is arbitrary then T is continuous on X.

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So T is continuous at $u_0 \in X$ and since u_0 is arbitrary then T is continuous on X.

Now suppose $T : X \to Y$ is continuous. Since T is linear then $T(0) = T(0 + 0) = T(0) + T(0)$ and so $T(0) = 0$. Let $\varepsilon = 1$. Since T is continuous at $u = 0$ then there is $\delta > 0$ such that if $\|u - 0\| < \delta$ then $\|T(u) - T(0)\| = \|T(u)\| < 1$.

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Theorem 13.1 (continued)

Theorem 13.1 A linear operator between normed linear spaces is continuous if and only if it is bounded.

Proof (continued). For any $u \in X$ where $u \neq 0$, let $\lambda = \delta/(2\|u\|)$. So $\|\lambda u\| = |\lambda|\|u\| = \lambda\|u\| = \delta/2 < \delta$. Thus $\|T(\lambda u)\| < 1$. Since $\|T(\lambda u)\| = \|\lambda T(u)\| = \lambda\|T(u)\|\delta/(2\|u\|) < 1$ then $\|T(u)\| = 2\|u\|/\delta - (2/\delta)\|u\|$ and so $\|T(u)\| \leq M\|u\|$ where $M = 2/\delta$ for all $u \in X$. Therefore, T is bounded. $\square$

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Proposition 13.2

Proposition 13.2. Let X and Y be normed linear spaces. Then the collection of bounded linear operators from X to Y, $\mathcal{L}(X, Y)$, is itself a normed linear space.

Proof. Let $T, S \in \mathcal{L}(X, Y)$. Then

$$\begin{aligned} \|(T+S)(u)\| &= \|T(u)+S(u)\| \\ &\leq \|T(u)\| + \|S(u)\| \text{ by the Triangle Inequality on } \mathbb{R} \\ &\leq \|T\|\|u\| + \|S\|\|u\| \text{ by the definition of operator norm} \\ &= (\|T\| + \|S\|)\|u\| \end{aligned}$$

and so $T + S$ is bounded by $\|T\| + \|S\|$. So $\mathcal{L}(X, Y)$ is closed under addition.

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and so $T+S$ is bounded by $\|T\|+\|S\|$. So $\mathcal{L}(X, Y)$ is closed under addition. For $\alpha \in \mathbb{R}$, $\|\alpha T(u)\| = |\alpha|\|T(u)\|$ by definition of $\|\cdot\|$ and so by Exercise 13.11,

$$\|\alpha T\| = \sup\{\|\alpha T(u)\| \mid u \in X, \|u\| \leq 1\}$$

$$= \sup\{|\alpha|\|T(u)\| \mid u \in X, \|u\| \leq 1\} = |\alpha| \sup\{\|T(u)\| \mid u \in X, \|u\| \leq 1\}$$

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Proof (continued). Finally, $\|T\| = 0$ means $\|T(u)\| \leq 0\|u\| = 0$ and so $T(u) = 0$ for all $u \in X$. If $T(u) = 0$ for all $u \in X$ then $\|T\| = \sup\{\|T(u)\| \mid u \in X, \|u\| \leq 1\} = 0$. Hence $\|T\| = 0$ if and only if $T = 0$. Therefore, $\|\cdot\|$ is a norm on $\mathcal{L}(X, Y)$ an d$\mathcal{L}(X, Y)$ is a normed linear space. □

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Theorem 13.3

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Proof. Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(X, Y)$. Let $u \in X$. For all $m, n \in \mathbb{N}$, $\|T_n(u) - T_m(u)\| = \|(T_n - T_m)(u)\| \leq \|T_n - T_m\|\|u\|$. So $\{T_n(u)\}$ is a Cauchy sequence in Y.

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$$T(u_1) + T(u_2) = \lim_{n\to\infty} T_n(u_1) + \lim_{n\to\infty} T_n(u_2)$$

$$= \lim_{n\to\infty} (T_n(u_1) + T_n(u_2)) = \lim_{n\to\infty} T_n(u_1 + u_2) = T(u_1 + u_2).$$

Similarly, for $u \in X$ and $\lambda \in \mathbb{R}$,

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Proof (continued). Combining these two results gives $T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$ and so T is linear. Let $\varepsilon > 0$. Since $\{T_n\}$ is Cauchy in $\mathcal{L}(X, Y)$, choose $N \in \mathbb{N}$ such that for $n \geq N$ and $k \geq 1$ we have $\|T_n - T_{n+k}\| < \varepsilon/2$. Then for all $u \in X$,

$$\|T_n(u) - T_{n+k}(u)\| = \|(T_n - T_{n+k})(u)\| \leq \|T_n - T_{n+k}\|\|u\| < \varepsilon\|u\|/2.$$

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