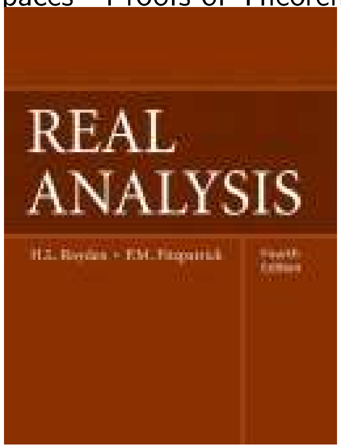
Real Analysis

Chapter 13. Continuous Linear Operators on Hilbert Between Banach Spaces

13.3. Compactness Lost: Infinite Dimensional Normed Linear Spaces—Proofs of Theorems



Theorem 13.4

Theorem 13.4. Any two norms on a finite dimensional linear space are equivalent.

Proof. Notice that the equivalence of norms is an equivalence relation on the set of norms on a set X (i.e., is reflexive, symmetric, and transitive). So the result follows if we choose a particular norm and show that any other norm is equivalent to the particular norm.

Let $\dim(X) = n$ and let $\{e_1, e_2, \ldots, e_n\}$ be a basis for Z. For any $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \in X$ define $\|x\|_* = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Then $\|\cdot\|_*$is the Euclidean norm on $\mathbb{R}^n$. Let $\|\cdot\|$ be any norm on X. Let $M = \max_{1\le i\le n} \|e_i\|$ and $c_2 = M\sqrt{n}$.

Theorem 13.4 (continued 1)

Proof (continued). Then

$$\|x\| = \|x_1e_2 + x_2e_2 + \cdots x_ne_n\| \le \sum_{i=1}^n |x_i|\|e_i\|$$

by the Triangle Inequality and positive homogeneity of $\|\cdot\|$

$$\le M \sum_{i=1}^n |x_i|.$$

By the Cauchy-Schwarz Inequality on $\mathbb{R}^n$, $|x\cdot y| \le \|x\|_*\|y\|_*$. For $i = 1, 2, \ldots, n$ define $\begin{cases} 1 \text{ if } x_i \ge 0 \\ -1 \text{ if } x_i < 0. \end{cases}$ Then $x_iy_i = |x_i|$ and

$$|x\cdot y| = \left|\sum_{i=1}^n x_iy_i\right| = \left|\sum_{i=1}^n |x_i|\right| = \sum_{i=1}^n |x_i| \le \|x\|_*\|y\|_* = \sqrt{n}\|x\|_*.$$

So from above we have $\|x\| \le M\sum_{i=1}^n |x_i| \le M\sqrt{n}\|x\|_* = c_2\|x\|_*$.

Theorem 13.4 (continued 2)

Proof (continued). Now define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, x_2, \ldots, x_n) = \|\sum_{i=1}^n x_ie_i\| = \|x\|$. Notice that for any $u, v \in X$, since $\|\cdot\|$ is a norm, then $|\|u\| - \|v\|| \le \|u - v\|$ (this follows from the Triangle Inequality). So for any $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, we have

$$|f(x_1, x_2, \ldots, x_n) - f(y_1, y_2, \ldots, y_n)| = \left|\left\|\sum_{i=1}^n x_ie_i\right\| - \left\|\sum_{i=1}^n y_ie_i\right\|\right|$$

$$\le \left\|\sum_{i=1}^n x_ie_i - \sum_{i=1}^n y_ie_i\right\| = \|x - y\| \le c_2\|x - y\|_*.$$

So $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz (where $\mathbb{R}^n$ has the metric induced by the Euclidean norm and $\mathbb{R}$ has the metric induced by absolute value). Since f is Lipschitz then it is continuous (by Exercise 13.9(ii)). Since $\{e_1, e_2, \ldots, e_n\}$ is a basis then it is linearly independent. So if we take $x = (x_1, x_2, \ldots, x_n) \in S = \{x \in \mathbb{R}^n \mid \sum_{i=1}^m x_i^2 = 1\} \subset \mathbb{R}^n$ then $f(x) > 0$ (norm $\|\cdot\|$ is zero only for the zero vector, by definition of "norm").

Theorem 13.4 (continued 3)

Proof (continued). Let S is closed and bounded in $\mathbb{R}^n$ and so by the Heine-Borel Theorem (Theorem 9.20) S is compact. A continuous real valued function on a compact set takes on a minimum value by the Extreme Value Theorem (Theorem 9.22), say $m > 0$. So for any $(x_1, x_2, \ldots, x_n) \in S \subset \mathbb{R}^n$ we have $f(x_1, x_2, \ldots, x_n) = \|\sum_{i=1}^n x_i e_i\| = \|x\| \geq m$. So for any $x' \in X$, say $x' = \sum_{i=1}^n x_i' e_i$, let $N^2 = \sum_{i=1}^n (x_i)^2$ so that $\sum_{i=1}^m (x_i'/N)^2 = 1$ and $(x_1'/N, x_2'/N, \ldots, x_n'/N) \in S$. Then

$$\|x'\| = N\|x'/N\| \geq Nm = \|(x_1', x_2', \ldots, x_n')\|_* m = m\|x'\|_*.$$

That is, $\|x\|_* \leq (1/m)\|x\|$ for all $x \in X$.

So we have constants $c_1 = m$ and c_2 such that for all $x \in X$, $c_1\|x\|_* \leq \|x\| \leq c_2\|x\|_*$. Therefore $\|\cdot\|_*$ and $\|\cdot\|$ are equivalent and hence any two norms on $\mathbb{R}^n$ are equivalent. □

Corollary 13.5

Corollary 13.5. Any two normed linear spaces of the same finite dimension are isomorphic.

Proof. Notice that isomorphisms of normed linear spaces from an equivalence relation on the set of all normed linear spaces. So the result follows if we choose a particular normed linear space and show that any other normed linear space is isomorphic to the particular normed linear space.

Let n be the common dimension and consider $\mathbb{R}^n$ under the Euclidean norm. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for X. Define $T : \mathbb{R}^n \to X$ by mapping $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ to $T(x) = \sum_{i=1}^n x_i e_i \in X$. Since $\{e_1, e_2, \ldots, e_n\}$ is a basis then T is one to one (by the "uniqueness" property of a basis) and onto (by the "spanning" property of a basis). T is linear. We need to show that T and T^{-1} are continuous. Since a linear operator is continuous if and only if it is bounded by Theorem 13.1, we only need to show boundedness.

Corollary 13.5 (continued 1)

Proof (continued). For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ define $\|x\|' = \|T(x)\| = \|\sum_{i=1}^n x_i e_i\|$ where $\|\cdot\|$ is a norm on X. Since T is linear then $T(0) = 0$ and since T is one to one $T(x) = 0$ if and only if $x = 0$. So $\|x\|' = \|T(x)\| = 0$ if and only if $x = 0$ and nonnegativity holds for $\|\cdot\|'$. For $\alpha x \in \mathbb{R}^n$,

$$\|\alpha x\|' = \|T(\alpha x)\| = \left\|\sum_{i=1}^n (\alpha x_i) e_i\right\|$$

$$= \left\|\alpha \sum_{i=1}^n x_i e_i\right\| = |\alpha| \left\|\sum_{i=1}^n x_i e_i\right\| = |\alpha|\|T(x)\| = |\alpha|\|x\|'$$

and so $\|\cdot\|'$ satisfies positive homogeneity. For $x, y \in \mathbb{R}$, $\|x+y\|' = \|T(x+y)\| = \|T(x)+T(y)\| \leq \|T(x)\|+\|T(y)\| = \|x\|'+\|y\|'$ and so $\|\cdot\|'$ satisfies the Triangle Inequality. Therefore $\|\cdot\|'$ is a norm on $\mathbb{R}^n$.

Corollary 13.5 (continued 2)

Corollary 13.5. Any two normed linear spaces of the same finite dimension are isomorphic.

Proof. By Theorem 13.4, $\|\cdot\|'$ is equivalent to the Euclidean norm $\|\cdot\|_*$, so there are $c_1 \geq 0$, $c_2 \geq 0$ with $c_1\|x\|_* \leq \|T(x)\| \leq c_2\|x\|_*$ for all $x \in \mathbb{R}^n$. So $T : \mathbb{R}^n \to X$ is bounded (namely, $\|T\| \leq c_2$). For $y \in X$ where $y = T(x)$ for $x \in \mathbb{R}^n$ we have $c_1\|x\|_* \leq \|T(x)\|$ or (since $x = T^{-1}(y)$) $c_1\|T^{-1}(y)\|_* \leq \|y\|$; that is, $\|T^{-1}(y)\|_* \leq (1/c_1)\|y\|$ (notice in the proof of Theorem 13.4 that $c_1 > 0$) and so T^{-1} is bounded. Therefore, T and T^{-1} are continuous and $T : \mathbb{R}^n \to X$ is an isomorphism. Hence any two dimension-n normed linear spaces are isomorphic to $\mathbb{R}^n$ and hence are isomorphic to each other. □

Corollary 13.6

Corollary 13.6. Any finite dimensional normed linear space is complete and therefore any finite dimensional subspace of a normed linear space is closed.

Proof. Let X be a dimension-n normed linear space. Then by Corollary 13.5, X is isomorphic to $\mathbb{R}^n$. $\mathbb{R}^n$ is complete by Theorem 9.12(i) (this follows from the completeness of $\mathbb{R}$). Completeness is preserved under normed linear space isomorphisms (continuity of the isomorphism and its inverse gives that a sequence is Cauchy and convergent in one space if and only if its image is Cauchy and convergent in the other space). So X is complete.

Let Y be a finite dimensional subspace of a normed linear space X. Then Y is complete by the previous paragraph. By Proposition 9.11, Y is (topologically) closed. □

Corollary 13.7

Corollary 13.7. The closed unit ball in a finite dimensional normed linear space is compact.

Proof. Let X be a normed linear space of dimension n and B be its closed unit ball. By Corollary 13.5, X is isomorphic to $\mathbb{R}^n$, so let $T : X \to \mathbb{R}^n$ be an isomorphism (so, by the definition of "isomorphism," T and T^{-1} are continuous). Since B is a bounded set in X and T is a bounded transformation by Theorem 13.1, then $T(B)$ is bounded (namely, by $\|T\| \cdot 1$). So $T(B)$ is bounded and closed since T^{-1} is continuous (by Proposition 9.8, inverse images of open sets under a continuous function are open and similarly for closed sets). So $T(B)$ is a closed and bounded subset of $\mathbb{R}^n$ and hence by Theorem 9.20, $T(B)$ is compact. Since T^{-1} is continuous then $T^{-1}(T(B)) = B$ is compact by Proposition 9.21, as claimed. □

Riesz's Lemma

Riesz's Lemma. Let Y be a (topologically) closed proper linear subspace of a normed linear space X. Then for each $\varepsilon > 0$ there is a unit vector $x_0 \in X$ for which $\|x_0 - y\| > 1 - \varepsilon$ for all $y \in Y$.

Proof. We give the proof for the case $\varepsilon = 1/2$ and leave the general case as Exercise 13.24. We only need the result for $\varepsilon = 1/2$ to prove the Riesz Theorem. Since Y is a proper subset of X then there is $x \in X \setminus Y$. Since Y is a (topologically) closed subset of X then $X \setminus Y$ is open and so there is a ball of some positive radius centered at x that is disjoint from Y. So if we take infimum of the distance from x to an element of Y then we get a distance $d > 0$: $\inf\{\|x - y'\| \mid y' \in Y\| = d > 0$. Choose $y_1 \in Y$ for which $\|x - y_1\| < 2d$ (which can be done by the infimum definition of d). Define $x_0 = (x - y_1)/\|x - y_1\|$. Then for any $y \in Y$,

$$x_0 - y = \frac{x - y_1}{\|x - y_1\|} - y = \frac{x - y_1 - y\|x - y_1\|}{\|x - y_1\|} = \frac{1 - y'}{\|x - y_1\|}$$

where $y' = y_1 + y\|x - y_1\| \in Y$.

Riesz's Lemma (continued)

Riesz's Lemma. Let Y be a (topologically) closed proper linear subspace of a normed linear space X. Then for each $\varepsilon > 0$ there is a unit vector $x_0 \in X$ for which $\|x_0 - y\| > 1 - \varepsilon$ for all $y \in Y$.

Proof (continued). Then

$$\begin{aligned} \|x_0 - y\| &= \frac{\|x - y'\|}{\|x - y_1\|} \text{ since } x_0 - y = (x - y')/\|x - y_1\| \\ &> \|x - y'\|/(2d) \text{ since } \|x - y_1\| < 2d \\ &\geq d/(2d) \text{ since } d \leq \|x - y'\| \text{ by the choice of } d \\ &= 1/2. \end{aligned}$$

Since $y \in Y$ is arbitrary, the result follows. □

Riesz'z Theorem

Riesz's Theorem. The closed unit ball of a normed linear space X is compact if and only if X is finite dimensional.

Proof. If X is finite dimensional then the closed unit ball is compact by Corollary 13.7.

Assume $\dim(X) = \infty$. We show that X is not sequentially compact. Let $x_1 \in B$. Consider the space of $\{x_1\}$, $X_1 = \{x \in X \mid x = \alpha x_1 \text{ for some } \alpha \in \mathbb{R}\}$. Then X_1 is a closed proper linear subspace of X ("closed" because any convergent sequence of elements of X_1 converges to an element of X_1, so that X_1 contains all of its limit points; "proper" because $\dim(X) = \infty$). So by Riesz's Lemma with $\varepsilon = 1/2$, there is $x_2 \in B$ for which $\|x_1 - x_2\| > 1/2$. We now use induction. Suppose we have chosen n vectors in B, $\{x_1, x_2, \ldots, x_n\}$, each pair of which are more than a distance $1/2$ apart. Let X_n be the span of $\{x_1, x_2, \ldots, x_n\}$.

Riesz'z Theorem (continued)

Riesz's Theorem. The closed unit ball of a normed linear space X is compact if and only if X is finite dimensional.

Proof (continued). Then X_n is a finite dimensional subspace of X and is closed by Corollary 13.6, and X_n is a proper subspace of X since $\dim(X) = \infty$. Again, by Riesz's Lemma with $\varepsilon = 1/2$, there is $x_{n+1} \in B$ for which $\|x_i - x_{n+1}\| > 1/2$ for $1 \le i \le n$. The resulting sequence $(x_i) \subset B$ satisfies $\|x_n - x_m\| > 1/2$ for any $n \ne m$, so it has no Cauchy subsequence and therefore no convergent subsequences (recall that convergent sequences are always Cauchy by the Triangle Inequality). So B is not sequentially compact (by the definition) and by Theorem 9.16, B is not compact (here we treat B as a metric space). □