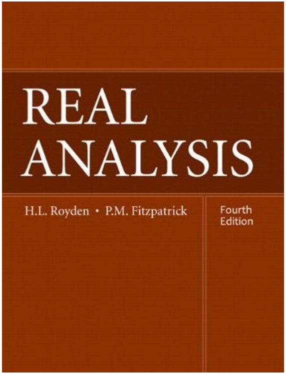
Real Analysis

Chapter 14. Duality for Normed Linear Spaces

14.1. Linear Functionals, Bounded Linear Functionals, and Weak Topologies—Proofs of Theorems



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Lemma 14.1.A

Lemma 14.1.A. Let X be a linear space and $\psi \in X^\sharp$, $\psi \neq 0$, and $x_0 \in X$ for which the direct sum $X = (\text{Ker}(\psi)) \oplus \text{span}\{x_0\}$, where $\text{Ker}(\psi) = \{x \in X \mid \psi(x) = 0\}$.

Proof. Since $\psi(x_0) \neq 0$, then $(\text{Ker}(\psi)) \cap \text{span}\{x_0\} = \{0\}$. For $x \in X$ we have

$$x = \left(x - \frac{\psi(x)}{\psi(x_0)}x_0\right) + \frac{\psi(x)}{\psi(x_0)}x_0$$

where $(\psi(x)/\psi(x_0))x_0 \in \text{span}\{x_0\}$ and

$$\psi\left(x - \frac{\psi(x)}{\psi(x_0)}x_0\right) = \psi(x) - \frac{\psi(x)}{\psi(x_0)}\psi(x_0) = 0$$

so that $x - (\psi(x)/\psi(x_0))x_0 \in \text{Ker}(\psi)$. So $x \in (\text{Ker}(\psi)) \oplus \text{span}\{x_0\}$ and the claim follows. □

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Proposition 14.1. A linear subspace X_0 of a linear space X is of codimension 1 if and only if $X_0 = \text{Ker}(\psi)$ for some nonzero $\psi \in X^\sharp$.

Proof. By Lemma 14.1.A and the definition of "codimension 1," we have that the kernel of a linear functional is of codimension 1. For the converse, suppose X_0 is a subspace of codimension 1. Then there is $x_0 \neq 0$ for which $X = X_0 \oplus \text{span}\{x_0\}$ by the definition of "codimension 1."

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$$\psi(x+y) = \psi((x_1 + \lambda_x x_0) + (y_1 + \lambda_y x_0))$$

$$= \psi((x_1 + y_1) + (\lambda_x + \lambda_y)x_0) = \lambda_x + \lambda_y = \psi(x) + \psi(y)$$

and $\psi \in X^\sharp$. Finally, $\text{Ker}(\psi) = \{x \in X \mid x = x_1 + 0x_0, x_1 \in X_0\} = X_0$. □

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Proposition 14.2. Let Y be a linear subspace of a linear space X. Then each linear functional on Y is an extension to a linear functional on all of X. In particular, for each nonzero $x \in X$ there is a $\psi \in X^\sharp$ for which $\psi(x) \neq 0$.

Proof. Since Y is a subspace of X, by Exercise 13.36 (which requires Zorn's Lemma when $\dim(X) = \infty$) there is a linear subspace X_0 of X (called the *linear complement* of Y) such that $X = Y \oplus X_0$.

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$$\eta(x_1 + x_2) = \eta((y_1 + x_{01}) + (y_2 + x_{02})) = \eta((y_1 + y_2) + (x_{01} + x_{02}))$$

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Proof (continued). For the "in particular" part, let $x \in X$, $x \neq 0$, and define $\eta : \text{span}\{x\} \to \mathbb{R}$ by $\eta(\lambda x) = \lambda \|x\|$. Then

$$\eta(\lambda_1 x + \lambda_2 x) = \eta((\lambda_1 + \lambda_2)x) = (\lambda_1 + \lambda_2)\|x\|$$

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Suppose X is infinite dimensional. Let $\mathcal{B}$ be a Hamel basis for X. We can normalize the vectors of $\mathcal{B}$, so without loss of generality we can assume the vectors of $\mathcal{B}$ are unit vectors. Since $\mathcal{B}$ is infinite, we may "choose" a countable infinite subset of $\mathcal{B}$, $\{x_k\}_{k=1}^\infty$ (every infinite set has a countable infinite subset).

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$$\psi(x+y) = \sum_{k=1}^{\infty} k\psi_k(x+y) = \sum_{k=1}^{\infty} (k\psi_k(x) + k\psi_k(y)).$$

Since $\mathcal{B}$ is a Hamel basis, then each $x \in X$ is a finite linear combination of the elements of $\mathcal{B}$ and so only finitely many of the $\psi_k(x)$ are nonzero. So the "series" $\sum_{k=1}^{\infty}(k\psi_k(x) + k\psi_k(y))$ converges absolutely and hence

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Proof. If ψ is a linear combination of the $\{\psi_i\}_{i=1}^n$ then for $x \in \cap_{i=1}^n \text{Ker}(\psi_i)$, certainly $x \in \text{Ker}(\psi)$.

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Proposition 14.4 (continued 1)

Proof (continued). Let $\lambda_1 = 1/\psi_1(x_0)$. Then for $x \in X$,

$$\lambda_1\psi_1(x) = \lambda_1\psi_1(x' + \alpha_x x_0) = \lambda_1\psi_1(x') + \lambda_1\alpha_x\psi_1(x_0)$$

$$= 0 + \alpha_x\psi_1(x_0)/\psi_1(x_0) = \alpha_x = \psi(x).$$

So $\psi - \lambda_1\psi_1$, ψ is a linear combination of $\{\psi_i\|_{i=1}^1$ and the result holds for $n = 1$.

Now assume the result holds for $n = k - 1$ and ψ is a linear combination of $\{\psi_i\}_{i=1}^{k-1}$. Suppose the hypothesis holds for $n = k$, $\cap_{i=1}^k \text{Ker}(\psi_i) \subset \text{Ker}(\psi)$. If $\psi_k = 0$ then the result holds and ψ is a linear combination of $\{\psi_i\}_{i=1}^k \text{Ker}(\psi_i) \subset \text{Ker}(\psi)$. If $\psi_k = 0$ then the result holds and ψ is a linear combination of $\{\psi_i\}_{i=1}^k$. So without loss of generality, there is $x_0 \in X$ with $\psi_k(x_0) = 1$.

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$$\cap_{i=1}^{k}\text{Ker}(\psi_i) = \cap_{i=1}^{k-1}(\text{Ker}(\psi_i) \cap Y) \subset \text{Ker}(\psi) \cap Y.$$

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$$= 0 + \alpha_x\psi_1(x_0)/\psi_1(x_0) = \alpha_x = \psi(x).$$

So $\psi - \lambda_1\psi_1$, ψ is a linear combination of $\{\psi_i\|_{i=1}^1$ and the result holds for $n = 1$.

Now assume the result holds for $n = k - 1$ and ψ is a linear combination of $\{\psi_i\}_{i=1}^{k-1}$. Suppose the hypothesis holds for $n = k$, $\cap_{i=1}^k \mathrm{Ker}(\psi_i) \subset \mathrm{Ker}(\psi)$. If $\psi_k = 0$ then the result holds and ψ is a linear combination of $\{\psi_i\}_{i=1}^k \mathrm{Ker}(\psi_i) \subset \mathrm{Ker}(\psi)$. If $\psi_k = 0$ then the result holds and ψ is a linear combination of $\{\psi_i\}_{i=1}^k$. So without loss of generality, there is $x_0 \in X$ with $\psi_k(x_0) = 1$. Then by Lemma 14.1.A, $X = T \oplus \mathrm{span}\{x_0\}$ where $Y = \mathrm{Ker}(\psi_k)$ and so

$$\cap_{i=1}^k \mathrm{Ker}(\psi_i) = \cap_{i=1}^{k-1}(\mathrm{Ker}(\psi_i) \cap Y) \subset \mathrm{Ker}(\psi) \cap Y.$$

Proposition 14.4 (continued 2)

Proof (continued). By the induction assumption, there are $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ for which $\psi = \sum_{i=1}^{k-1} \lambda_i \psi_i$. For $x \in X$ we have $x = x' + \alpha_x x_0$ where $x' \in Y$, and so $\psi(x) = \psi(x') + \alpha_x \psi(x_0) = 0 + \alpha_x \psi(x_0)$. Let $\lambda_k = \psi(x_0) - \sum_{i=1}^{k-1} \lambda_i \psi_i(x_0)$ then for $x \in X$,

$$\begin{aligned}
\sum_{i=1}^{k} \lambda_i \psi_i(x) &= \sum_{i=1}^{k} \lambda_i \psi_i(x' + \alpha_x x_0) = \sum_{i=1}^{k} \lambda_i \psi_i(x') + \alpha_x \sum_{k=1}^{k} \lambda_i \psi_i(x_0) \\
&= 0 + \alpha_x \left(\sum_{k=1}^{k-1} \lambda_i \psi_i(x_0) + \lambda_k \psi_k(x_0) \right) \\
&= 0 + \alpha_x \left(\sum_{k=1}^{k-1} \lambda_i \psi_i(x_0) + \left(\psi(x_0) - \sum_{i=1}^{k-1} \lambda_i \psi_i(x_0) \right) \psi_k(x_0) \right) \\
&= \alpha_x \psi(x_0) \text{ since } \psi_k(x_0) = 1 \\
&= \psi(x).
\end{aligned}$$

Proposition 14.4 (continued 2)

Proof (continued). By the induction assumption, there are $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ for which $\psi = \sum_{i=1}^{k-1} \lambda_i \psi_i$. For $x \in X$ we have $x = x' + \alpha_x x_0$ where $x' \in Y$, and so $\psi(x) = \psi(x') + \alpha_x \psi(x_0) = 0 + \alpha_x \psi(x_0)$. Let $\lambda_k = \psi(x_0) - \sum_{i=1}^{k-1} \lambda_i \psi_i(x_0)$ then for $x \in X$,

$$\begin{aligned}
\sum_{i=1}^{k} \lambda_i \psi_i(x) &= \sum_{i=1}^{k} \lambda_i \psi_i(x' + \alpha_x x_0) = \sum_{i=1}^{k} \lambda_i \psi_i(x') + \alpha_x \sum_{k=1}^{k} \lambda_i \psi_i(x_0) \\
&= 0 + \alpha_x \left(\sum_{k=1}^{k-1} \lambda_i \psi_i(x_0) + \lambda_k \psi_k(x_0) \right) \\
&= 0 + \alpha_x \left(\sum_{k=1}^{k-1} \lambda_i \psi_i(x_0) + \left(\psi(x_0) - \sum_{i=1}^{k-1} \lambda_i \psi_i(x_0) \right) \psi_k(x_0) \right) \\
&= \alpha_x \psi(x_0) \text{ since } \psi_k(x_0) = 1 \\
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\end{aligned}$$

Proposition 14.4 (continued 3)

Proposition 14.4. Let X be a linear space, let $\psi \in X^\sharp$ and $\{\psi_i\}_{i=1}^n \subset X^\sharp$. Then ψ is a linear combination of $\{\psi_i\}_{i=1}^n$ if and only if $\cap_{i=1}^n \text{Ker}(\psi_i) \subset \text{Ker}(\psi)$.

Proof (continued). So $\psi = \sum_{i=1}^k \lambda_i \psi_i$, ψ is a linear combination of $\{\psi_i\}_{i=1}^k$ and the result holds for $n = k$. Therefore, by mathematical induction, the result holds for all $n \in \mathbb{N}$. □

Proposition 14.5

Proposition 14.5. Let X be a linear space and W a subspace of $X^\sharp$. Then a linear functional $\psi : X \to \mathbb{R}$ is E-weakly continuous if and only if it belongs to W.

Proof. By the definition of the W-weak topology, each linear functional in W is W-weakly continuous.

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For the converse, suppose $\psi : X \to \mathbb{R}$ is W-weakly continuous. Since ψ is W-weakly continuous at 0, there is a neighborhood $\mathcal{N}$ of 0 for which $|\psi(x)| = |\psi(x) - \psi(0)| < 1$ if $x \in \mathcal{N}$ (by the definition of continuity with $\varepsilon = 1$).

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Lemma 14.1.B

Lemma 14.1.B. The evaluation functional $J(x)$ is linear and bounded. That is, $J(x) \in (X^*)^*$.

Proof. For $\psi_1, \psi_2 \in X^*$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ we have

$$J(x)[\alpha_1 \pi_1 + \alpha_2 \psi_2] = (\alpha_1 \psi_1 + \alpha_2 \psi_2)(x)$$

$$= \alpha_1 \psi_1(x) + \alpha_2 \psi_2(x) = \alpha_1 J(x)[\psi_1] + \alpha_2 J(x)[\psi_2],$$

so $J(x)$ is linear.

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so $J(x)$ is linear. $J(x)$ is bounded because for any $\psi \in X^*$, $|J(x)[\psi]| = |\psi(x)\| \leq \|\psi\||x|$ and so $\|J(x)\| \leq |x|$. □

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Proposition 14.6

Proposition 14.6. A normed linear space X is reflexive if and only if the weak and weak-$*$ topologies are the same.

Proof. By definition, X is reflexive if $J(X) = X^{**}$. So if X is reflexive, the topology induced by $J(X)$ (the weak-$*$ topology on X^*) is the same as the topology induced by X^{**} (the weak topology on X^*).

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Conversely, suppose the weak and weak-$*$ topologies are the same. Let $\Psi : X^* \to \mathbb{R}$ be a linear functional continuous with respect to the norm on X; that is, let $\Psi \in X^{**}$. By definition of the weak topology, Ψ is continuous with respect to the weak topology on X^*.

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