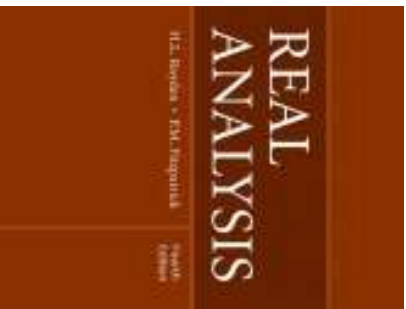


Chapter 14. Duality for Normed Linear Spaces

14.2. The Hahn-Banach Theorem—Proofs of Theorems



The Hahn-Banach Lemma

The Hahn-Banach Lemma. Let p be a positively homogeneous, subadditive functional on the linear space X and Y a subspace of X on which there is defined a linear functional ψ for which $\psi \leq p$ on Y . Let z belong to $X \setminus Y$. Then ψ can be extended to a linear functional ψ on $\text{span}[Y + z]$ for which $\psi \leq p$ on $\text{span}[Y + z]$.

Proof. Since $x \notin X \setminus Y$, then every vector in $\text{span}[Y + z]$ can be written uniquely as $y + \lambda z$ for $y \in Y$ and $\lambda \in \mathbb{R}$ (if $y_1 + \lambda_1 z = y_2 + \lambda_2 z$ then $(\lambda_1 - \lambda_2)z \in Y$, but $z \notin Y$ and Y a linear space implies that $\lambda_1 - \lambda_2 = 0$ and $\lambda_1 = \lambda_2$; it follows that $y_1 = y_2$). We extend ψ from Y to $\text{span}[Y + z]$ by defining $\psi(y + \lambda z) = \psi(y) + \lambda\psi(z)$ where the value of $\psi(z)$ is given below. So to show $\psi \leq p$ on $\text{span}[Y + z]$, it is sufficient to show that for all $y \in Y$ and all $\lambda \in \mathbb{R}$ we have

$$\psi(y + \lambda z) = \psi(y) + \lambda\psi(z) \leq p(y + \lambda z). \tag{9}$$

The Hahn-Banach Lemma (continued 1)

Proof (continued). We now choose a value for $\psi(z)$. For any vectors $y_1, y_2 \in Y$, since ψ is linear, $\psi \leq p$ on Y and p is subadditive, then

$$\begin{aligned} \psi(y_1) + \psi(y_2) &= \psi(y_1 + y_2) \leq p(y_1 + y_2) \\ &= p((y_1 - z) + (y_2 + z)) \leq p(y_1 - z) + p(y_2 + z). \end{aligned}$$

Since this holds for all y_1 and y_2 and there are only y_1 's on the left and only y_2 's on the right, then

$\sup\{\psi(y) = p(y - z)\} \leq \inf\{-\psi(y) + p(y + z)\}$ (notice that both of these are finite). Define $\psi(z) = \sup\{\psi(y) - p(y - z) \mid y \in Y\}$. Then for any $y \in Y$, $\psi(y) - p(y - z) \leq \psi(z) \leq -\psi(y) + p(y + z)$ (we could in fact define $\psi(z)$ to be any value between $\sup\{\psi(y) - p(y - z)\}$ and $\inf\{-\psi(y) + p(y + z)\}$).

Let $y \in Y$. For $\lambda > 0$, in the inequality $\psi(z) \leq -\psi(y) + p(y + z)$, replace y with y/λ to get $\psi(z) \leq -\psi(y/\lambda) + p(y/\lambda + z)$ or

$$\lambda\psi(z) \leq -\lambda\psi(y/\lambda) + \lambda p(y/\lambda + z) \text{ or } \lambda\psi(z) \leq -\psi(y) + p(y + \lambda z) \text{ or } \psi(y) + \lambda\psi(z) \leq p(y + \lambda z),$$

which is (9) and the result holds for $\lambda > 0$.

The Hahn-Banach Lemma (continued 2)

The Hahn-Banach Lemma. Let p be a positively homogeneous, subadditive functional on the linear space X and Y a subspace of X on which there is defined a linear functional ψ for which $\psi \leq p$ on Y . Let z belong to $X \setminus Y$. Then ψ can be extended to a linear functional ψ on $\text{span}[Y + z]$ for which $\psi \leq p$ on $\text{span}[Y + z]$.

Proof (continued). For $\lambda < 0$ in the inequality

$$\begin{aligned} \psi(-y/\lambda) - p(-y/\lambda - z) &\leq \psi(z) \text{ or} \\ \lambda\psi(-y/\lambda) + \lambda p(-y/\lambda - z) &\leq -\lambda\psi(z) \text{ or } \psi(y) - p(y + \lambda z) \leq -\lambda\psi(z) \text{ or} \\ \psi(y) + \lambda\psi(z) &\leq p(y + \lambda z) \text{ which is (0) and the result holds for } \lambda < 0. \text{ Of} \\ \text{course, (9) holds trivially for } \lambda &= 0. \text{ Hence, } \psi \text{ defined as} \\ \psi(y) + \lambda\psi(z) &= \psi(y + \lambda z) \leq p(y + \lambda z) \text{ on } \text{span}[Y + z] \text{ as claimed. } \square \end{aligned}$$

The Hahn-Banach Theorem

The Hahn-Banach Theorem. Let p be a positively homogeneous,

subadditive functional on a linear space X and Y a subspace of X on which there is defined a linear functional ψ for which $\psi \leq p$ on Y . Then ψ may be extended to a linear functional ψ on all of X for which $\psi \leq p$ on all of X .

Proof. Consider the family \mathcal{F} of all linear functionals η defined on a subspace Y_η of X for which $Y \subset Y_\eta$, $\eta = \psi$ on Y , and $\eta \leq p$ on Y_η .

Notice that $\psi \in \mathcal{F}$ where $Y_\psi = Y$ and so \mathcal{F} is nonempty. Partially order \mathcal{F} by defining $\eta \prec \eta_2$ if $Y_{\eta_1} \subset Y_{\eta_2}$ and $\eta_1 = \eta_2$ on Y_{η_1} .

To apply Zorn's Lemma, we need to show that every totally ordered subfamily of \mathcal{F} has an upper bound. Let \mathcal{F}_0 be a totally ordered subfamily of \mathcal{F} . Define Z to be the union of the domains of the functionals in \mathcal{F}_0 (that is, the union of the Y_η 's for $\eta \in \mathcal{F}_0$).

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The Hahn-Banach Theorem (continued 2)

The Hahn-Banach Theorem. Let p be a positively homogeneous,

subadditive functional on a linear space X and Y a subspace of X on which there is defined a linear functional ψ for which $\psi \leq p$ on Y . Then ψ may be extended to a linear functional ψ on all of X for which $\psi \leq p$ on all of X .

Proof (continued).

Zorn's lemma implies that \mathcal{F} has a maximal member ψ_0 . Let the domain of ψ_0 be Y_0 . By definition, $Y \subset Y_0$ and $\psi_0 \leq p$ on Y_0 . If there is some $z \in X \setminus Y_0$, then the Hahn-Banach Lemma implies there is a linear

functional η' defined on $\text{span}[Z + z]$ such that $\eta' = \eta_0^*$ on Z . But then $\eta_0^* \prec \eta'$, contradicting the maximality of η_0^* . So there is no such z and in fact $Z = X$. \square

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The Hahn-Banach Theorem (continued 1)

Proof (continued). Since the domains in \mathcal{F}_0 are nested (they form an increasing sequence of sets), then for any finite collection of vectors of Z , there is some domain containing all of them and since domains are linear spaces then this domain contains every linear combination of elements of Z are again in Z and therefore Z is a subspace of X . For $z \in Y$, choose $\eta \in \mathcal{F}_0$ such that $z \in Y_\eta$, and then define $\eta^*(z) = \eta(z)$. By the nestedness of the domains, η^* is well defined and since each η is linear on Y_η then (similar to the above argument showing Z is a subspace of X) η^* is linear in Z . Now $\eta^* \leq p$ on Z since each $\eta \leq p$. Also, $Y \subset Z$ and $\eta_0^* = Z$ and $\eta = \eta_0^*$ on Y_η for all $\eta \in \mathcal{F}_0$, then $\eta \prec \eta_0^*$ for all $\eta \in \mathcal{F}_0$. So arbitrary totally ordered subfamily \mathcal{F}_0 of \mathcal{F} has an upper bound, then Zorn's Lemma applies to \mathcal{F} .

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Theorem 14.7

Theorem 14.7. Let X_0 be a linear subspace of a normed linear space X .

Then each bounded linear functional ψ on X_0 has an extension to a bounded linear functional on all of X that has the same norm as ψ . In particular, for each $x \in X$ with $x \neq 0$ there is $\psi \in X^*$ for which $\psi(x) = \|x\|$ and $\|\psi\| = 1$.

Proof. Let $\psi : X_0 \rightarrow \mathbb{R}$ be linear and bounded. Define

$$M = \|\psi\| = \sup\{|\psi(x)| \mid x \in X_0, \|x\| \leq 1\}.$$
 Define $p : X \rightarrow \mathbb{R}$ by

$$p(x) = M\|x\| \text{ for all } x \in X. \text{ Then } p(\lambda x) = M\|\lambda x\| = M\lambda\|x\| = \lambda p(x) \text{ for}$$

all $\lambda > 0$ and $p(x+y) = M\|x+y\| \leq M\|x\| + M\|y\| = p(x) + p(y)$, so p is positively homogeneous and subadditive. Since $M = \|\psi\|$ then $\psi \leq p$ on X_0 .

By the Hahn-Banach Theorem, ψ can be extended to a continuous linear functional ψ defined on all of X and $\psi(x) \leq p(x) = M\|x\|$ for all $x \in X$. Replacing x with $-x$ gives $\psi(-x) \leq p(-x) = M\| -x \|$ or $-\psi(x) \leq -p(x) = M\|x\|$ or $-M\|x\| \leq p(x) \leq \psi(x)$ and hence $\|\psi(x)\| \leq p(x) = M\|x\|$ for all $x \in X$.

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Theorem 14.7 (continued)

Theorem 14.7. Let X_0 be a linear subspace of a normed linear space X .

Then each bounded linear functional ψ on X_0 has an extension to a bounded linear functional on all of X that has the same norm as ψ . In particular, for each $x \in X$ with $x \neq 0$ there is $\psi \in X^*$ for which $\psi(x) = \|x\|$ and $\|\psi\| = 1$.

Proof (continued). So the extension of ψ has the same bound on X by the Hahn-Banach Theorem. So the norm of the extension is at most M , but since $M = \sup\{|\psi(x)| \mid x \in X_0, \|x\| \leq 1\}$ then the norm of the extension is $\sup\{|\psi(x)| \mid x \in X, \|x\| \leq 1\} \geq M$. Therefore the norm of the extension equals $\|\psi\| = M$.

For the “in particular” part, let $x \in X$, $x \neq 0$. Define $\eta : \text{span}[x] \rightarrow \mathbb{R}$ by $\eta(\lambda x) = \lambda \|x\|$. Then $\|\eta\| = 1$. By the first part of the proof, functional η has an extension to a bounded linear functional on all of X that also has a norm of 1. □

Corollary 14.8 (continued)

Corollary 14.8. Let X be a normed linear space. If X_0 is a finite dimensional subspace of X , then there is a closed linear subspace X_1 of X for which $X = X_0 \oplus X_1$. That is, X_0 has a closed linear complement in X .

Proof (continued).

$$\psi'_k \left(x - \sum_{i=1}^n \psi'_i(x) e_i \right) = \psi'_k(x) - \sum_{i=1}^n \psi'_i(x) \psi'_k(e_i) = \psi'_k(x) - \psi'_k(x)(1) = 0,$$

so $x - \sum_{k=1}^n \psi'_k(x) e_k \in \bigcap_{k=1}^n \text{Ker}(\psi'_k)$. Therefore, $X = X_0 \oplus X_1$. □

Corollary 14.8

Corollary 14.8. Let X be a normed linear space. If X_0 is a finite dimensional subspace of X , then there is a closed linear subspace X_1 of X for which $X = X_0 \oplus X_1$. That is, X_0 has a closed linear complement in X .

Proof. Let e_1, e_2, \dots, e_n be a basis for X_0 . For $a \leq k \leq n$, define $\psi_k : X_0 \rightarrow \mathbb{R}$ by $\psi_k(\sum_{i=1}^n \lambda_i e_i) = \lambda_k$. Since X_0 is finite dimensional and each ψ_k is clearly linear then each ψ_k is continuous by Exercise 13.26. By Theorem 14.7 each ψ_k has an extension ψ'_k to all of X . Since ψ_k is continuous then it is bounded by Theorem 13.1. ψ'_k is bounded as given by Theorem 14.7, so ψ'_k is continuous by Theorem 13.1. Since $\psi'_k : X \rightarrow \mathbb{R}$ and \mathbb{R} is finite dimensional then by Exercise 13.2b, $\text{Ker}(\psi'_k)$ is closed in X for each $1 \leq k \leq n$. So subspace $X_1 = \bigcap_{k=1}^n \text{Ker}(\psi'_k)$ is closed in X .

The only element of X_0 in X_1 is 0. Also for $x \in X$ we have (similar to the proof of Lemma 14.1.A): $x = (\sum_{k=1}^n \psi'_k(x) e_k) + (x - \sum_{k=1}^n \psi'_k(x) e_k)$ where $\sum_{k=1}^n \psi'_k(x) e_k \in X_0$ and for each k

Corollary 14.9

Corollary 14.9. Let X be a normed linear space. Then the natural embedding $J : X \rightarrow X^{**}$ is an isometry.

Proof. Recall that by definition $J(x)[\psi] = \psi(x)$ for all $x \in X$ and $\psi \in X^*$. Let $x \in X$. Recall that by the definition of the operator norm we have $|\psi(x)| \leq \|\psi\| \|x\|$ for all $\psi \in X^*$. Thus $|J(x)[\psi]| = |\psi(x)| \leq \|x\| \|\psi\|$ for all $\psi \in X^*$. Therefore $J(x)$ is bounded and $\|J(x)\| \leq \|x\|$. By the “in particular” part of Theorem 14.7, there is $\psi \in X^*$ for which $\psi(x) = \|x\|$ and $\|\psi\| = 1$. So for this ψ , $J(x)[\psi] = \psi(x) = \|x\|$. So $\|x\| \leq \|J(x)\|$. Therefore $\|J(x)\| = \|x\|$ and so J is an isometry. □

Theorem 14.10

Theorem 14.10. Let X_0 be a subspace of the normed linear space X .

Then a point $x \in X$ belongs to the closure of X_0 if and only if whenever a functional $\psi \in X^*$ vanishes on X_0 , it also vanishes at x .

Proof. Let x be in the closure of X_0 . Then there is a sequence $\{x_n\} \subset X_0$ such that $\{x_n\} \rightarrow x$ by Proposition 9.6. Since ψ is bounded then it is continuous (Theorem 14.1) and so $\lim_{n \rightarrow \infty} \psi(x_n) = \psi(x)$, or $\psi(x) = 0$.

For the converse, let $x_0 \in X \setminus \overline{X_0}$. We need to show that there is $\psi \in X^*$ that vanishes on X_0 but $\psi(x_0) \neq 0$. Define $X = \overline{X_0} \oplus [x_0]$ and $\psi : Z \rightarrow \mathbb{R}$ by $\psi(z + \lambda x_0) = \lambda$ for all $x \in \overline{X_0}$ and $\lambda \in \mathbb{R}$. Notice that $\psi(x_0) = 1$. We need to show ψ is bounded and then we can apply Theorem 14.7. Since $\overline{X_0}$ is closed then $X \setminus \overline{X_0}$ is open. So there is $r > 0$ for which $\|u - x_0\| \geq r$ for all $u \in \overline{X_0}$ (negating the definition of "point of closure" in a metric space). □

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Corollary 14.11

Corollary 14.11. Let S be a subset of the normed linear space X . Then the linear span of S is dense in X if and only if whenever $\psi \in X^*$ vanishes on S , then $\psi = 0$.

Proof. Let $\text{span}[S]$ be dense in X . Then $\overline{\text{span}[S]} = X$ and so every point of X is a limit point of $\text{span}[S]$. If $\psi \in X^*$ vanishes on $\text{span}[S]$ then ψ vanishes on X by Theorem 14.10 and $\psi = 0$.

Suppose whenever $\psi \in X^*$ vanishes on $\text{span}[S]$ then $\psi = 0$. ASSUME $\text{span}[S]$ is not dense in X . Then there is some $x_0 \in X \setminus \overline{\text{span}[S]}$. By Theorem 14.10 there is some $\psi \in X^*$ vanishing on $\text{span}[S]$ but $\psi(x_0) \neq 0$, a CONTRADICTION. So the assumption that $\text{span}[S]$ is not dense in X is false and there $\text{span}[S]$ is dense in X . □

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Theorem 14.10 (continued)

Theorem 14.10. Let X_0 be a subspace of the normed linear space X .

Then a point $x \in X$ belongs to the closure of X_0 if and only if whenever a functional $\psi \in X^*$ vanishes on X_0 , it also vanishes at x .

Proof (continued). So for $x \in \overline{X_0}$ and $\lambda \in \mathbb{R}$,

$\|x + \lambda x_0\| = |\lambda| \|(-1/\lambda)x - x_0\| \geq |\lambda|r$, or $|\lambda| \leq \|x + \lambda x_0\|/r$. So $|\psi(x + \lambda x_0)| = |\lambda| \leq (1/r)\|x + \lambda x_0\|$ and hence $\|\psi\| \leq 1/r$, so that ψ has a bounded extension to all of X . The extension if in X^* , vanishes on X_0 (since ψ vanishes on X_0) and has a nonzero value at x_0 since $\psi(x_0) = 1 \neq 0$. □

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Theorem 14.12

Theorem 14.12. Let X be a normed linear space. Then every weakly convergent sequence in X is bounded. moreover, if $\{x_n\} \rightarrow x$ in X , then $\|x\| \leq \liminf \|x_n\|$.

Proof. Let $\{x_n\} \rightarrow x$ in X . Then, by the definition of weak convergence, $\lim_{n \rightarrow \infty} \psi(x_n) = \psi(x)$ for all $\psi \in X^*$. Recall that in Section 14.1 we defined $J(x) : X^* \rightarrow \mathbb{R}$ as $J(x)[\psi] = \psi(x)$ for all $\psi \in X^*$. So we define the sequence $\{J(x_n)\}$ of functionals mapping $X^* \rightarrow \mathbb{R}$ and we then have that this sequence of functionals converges to $J(x)$:

$$\lim_{n \rightarrow \infty} J(x_n)[\psi] = \lim_{n \rightarrow \infty} \psi(x_n) = \psi(x) = J(x)[\psi] \text{ for all } \psi \in X^*.$$

So for given ψ , $\{J(x_n)[\psi]\} \rightarrow J(x)[\psi]$ (since $J(x) : X^* \rightarrow \mathbb{R}$ the convergence is in \mathbb{R}). Every convergent sequence of real numbers is bounded, so there is some $M_\psi \geq 0$ such that $|J(x_n)[\psi]| \leq M_\psi$ for all $n \in \mathbb{N}$. So the family $\mathcal{F} = \{J(x_n) \mid n \in \mathbb{N}\} \subset \mathcal{L}(X^*, \mathbb{R})$ is pointwise bounded in the sense that for any given $\psi \in X^*$, $|J(x_n)[\psi]| \leq M_\psi$ for all $J(x_n) \in \mathcal{F}$.

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Theorem 14.12 (continued)

Theorem 14.12. Let X be a normed linear space. Then every weakly convergent sequence in X is bounded. moreover, if $\{x_n\} \rightharpoonup x$ in X , then $\|x\| \leq \liminf \|x_n\|$.

Proof (continued). Since \mathbb{R} is a Banach space, by Theorem 13.3 we have that $\mathcal{L}(X, \mathbb{R}) = X^*$ is a Banach space. So by the Uniform Boundedness Principle, there is a constant $M \geq 0$ for which $\|J(x_n)\| \leq M$ for all $n \in \mathbb{N}$. Since J is an isometry by Corollary 14.9, then the sequence $\{x_n\}$ is also bounded by M , as claimed.

For the “moreover” part, we know by Theorem 14.7 that there is a functional $\psi \in X^*$ for which $\|\psi\| = 1$ and $\psi(x) = \|x\|$. Then

$|\psi(x_n)| \leq \|\psi\| \|x_n\| = \|x_n\|$ for all $n \in \mathbb{N}$. Since $x_n \rightharpoonup x$ then $\psi(x_n) \rightarrow \psi(x)$; also $|\psi(x_n)| \rightarrow |\psi(x)| = \|x\|$. Therefore

$$\|x\| = \lim_{n \rightarrow \infty} |\psi(x_n)| \leq \liminf \|x_n\| \text{ since } |\psi(x_n)| \leq \|x_n\| \text{ for all } n \in \mathbb{N}. \quad \square$$