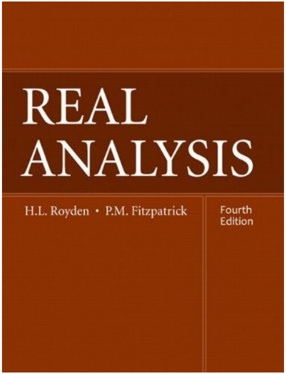
Real Analysis

Chapter 14. Duality for Normed Linear Spaces

14.2. The Hahn-Banach Theorem—Proofs of Theorems



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Proof. Since $x \notin X \setminus Y$, then every vector in $\text{span}[Y + z]$ can be written uniquely as $y + \lambda z$ for $y \in Y$ and $\lambda \in \mathbb{R}$ (if $y_1 + \lambda_1 z = y_2 + \lambda_2 z$ then $(\lambda_1 - \lambda_2)z \in Y$, but $z \notin Y$ and Y a linear space implies that $\lambda_1 - \lambda_2 = 0$ and $\lambda_1 = \lambda_2$; it follows that $y_1 = y_2$). We extend ψ from Y to $\text{span}[Y + z]$ by defining $\psi(y + \lambda z) = \psi(y) + \lambda\psi(z)$ where the value of $\psi(z)$ is given below.

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$$\psi(y + \lambda z) = \psi(y) + \lambda\psi(z) \leq p(y + \lambda z). \qquad (9)$$

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The Hahn-Banach Lemma (continued 1)

Proof (continued). We now choose a value for $\psi(z)$. For any vectors $y_1, y_2 \in Y$, since ψ is linear, $\psi \leq p$ on Y and p is subadditive, then

$$\psi(y_1) + \psi(y_2) = \psi(y_1 + y_2) \leq p(y_1 + y_2)$$
$$= p((y_1 - z) + (y_2 + z)) \leq p(y_1 - z) + p(y_2 + z).$$

Since this holds for all y_1 and y_2 and there are only y_1's on the left and only y_2's on the right, then $\sup\{\psi(y) = p(y - z)\} \leq \inf\{-\psi(y) + p(y + z)\}$ (notice that both of these are finite).

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$$\begin{aligned} \psi(y_1) + \psi(y_2) &= \psi(y_1 + y_2) \leq p(y_1 + y_2) \\ &= p((y_1 - z) + (y_2 + z)) \leq p(y_1 - z) + p(y_2 + z). \end{aligned}$$

Since this holds for all y_1 and y_2 and there are only y_1's on the left and only y_2's on the right, then $\sup\{\psi(y) = p(y - z)\} \leq \inf\{-\psi(y) + p(y + z)\}$ (notice that both of these are finite). Define $\psi(z) = \sup\{\psi(y) - p(y - z) \mid y \in Y\}$. Then for any $y \in Y$, $\psi(y) - p(y - z) \leq \psi(z) \leq -\psi(y) + p(y + z)$ (we could in fact define $\psi(z)$ to be any value between $\sup\{\psi(y) - p(y - z)\}$ and $\inf\{-\psi(y) + p(y + z)\}$).

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Let $y \in Y$. For $\lambda > 0$, in the inequality $\psi(z) \leq -\psi(y) + p(y + z)$, replace y with y/λ to get $\psi(z) \leq -\psi(y/\lambda) + p(y/\lambda + z)$ or $\lambda\psi(z) \leq -\lambda\psi(y/\lambda) + \lambda p(y/\lambda + z)$ or $\lambda\psi(z) \leq -\psi(y) + p(y + \lambda z)$ or $\psi(y) + \lambda\psi(z) \leq p(y + \lambda z)$, which is (9) and the result holds for $\lambda > 0$.

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Proof. Consider the family $\mathcal{F}$ of all linear functionals η defined on a subspace Y_η of X for which $Y \subset Y_\eta$, $\eta = \psi$ on Y, and $\eta \leq p$ on Y_n. Notice that $\psi \in \mathcal{F}$ where $Y_\psi = Y$ and so $\mathcal{F}$ is nonempty. Partially order $\mathcal{F}$ by defining $\eta \prec \eta_2$ if $Y_{\eta_1} \subset T_{\eta_2}$ and $\eta_1 = \eta_2$ on Y_{η_1}.

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To apply Zorn's Lemma, we need to show that every totally ordered subfamily of $\mathcal{F}$ has an upper bound. Let $\mathcal{F}_0$ be a totally ordered subfamily of $\mathcal{F}$. Define Z to be the union of the domains of the functionals in $\mathcal{F}_0$ (that is, the union of the Y_η's for $\eta \in \mathcal{F}_0$).

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Proof (continued). Since the domains in $\mathcal{F}_0$ are nested (they form an increasing sequence of sets), then for any finite collection of vectors of Z, there is some domain containing all of them and since domains are linear spaces then this domain contains every linear combination of elements of Z are again in Z and therefore Z is a subspace of X. For $z \in Y$, choose $\eta \in \mathcal{F}_0$ such that $z \in Y_\eta$, and then define $\eta^*(z) = \eta(z)$. By the nestedness of the domains, η^* is well defined and since each η is linear on Y_η then (similar to the above argument showing Z is a subspace of X) η^* is linear in Z. Now $\eta^* \leq p$ on Z since each $\eta \leq p$. Also, $Y \subset Z$ and $\eta^* = Z$ and $\eta = \eta^*$ on Y_η for all $\eta \in \mathcal{F}_0$, then $\eta \prec \eta^*$ for all $\eta \in \mathcal{F}_0$. So arbitrary totally ordered subfamily $\mathcal{F}_0$ of $\mathcal{F}$ has an upper bound, then Zorn's Lemma applies to $\mathcal{F}$.

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Theorem 14.7. Let X_0 be a linear subspace of a normed linear space X. Then each bounded linear functional ψ on X_0 has an extension to a bounded linear functional on all of X that has the same norm as ψ. In particular, for each $x \in X$ with $x \neq 0$ there is $\psi \in X^*$ for which $\psi(x) = \|x\|$ and $\|\psi\| = 1$.

Proof. Let $\psi : X_0 \to \mathbb{R}$ be linear and bounded. Define $M = \|\psi\| = \sup\{|\psi(x)| \mid x \in X_0, \|x\| \leq 1\}$. Define $p : X \to \mathbb{R}$ by $p(x) = M\|x\|$ for all $x \in X$.

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For the "in particular" part, let $x \in X$, $x \neq 0$. Define $\eta : \text{span}[x] \to \mathbb{R}$ by $\eta(\lambda x) = \lambda \|x\|$. Then $\|\eta\| = 1$. By the first part of the proof, functional η has an extension to a bounded linear functional on all of X that also has a norm of 1. □

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Corollary 14.8

Corollary 14.8. Let X be a normed linear space. If X_0 is a finite dimensional subspace of X, then there is a closed linear subspace X_1 of X for which $X = X_0 \oplus X_1$. That is, X_0 has a closed linear complement in X.

Proof. Let $e_1, e_2, \ldots, e_n$ be a basis for X_0. For $a \leq k \leq n$, define $\psi_k : X_0 \to \mathbb{R}$ by $\psi_k \left(\sum_{i=1}^{n} \lambda_i e_i \right) = \lambda_k$. Since X_0 is finite dimensional and each ψ_k is clearly linear then each ψ_k is continuous by Exercise 13.26.

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Proof. Let $e_1, e_2, \ldots, e_n$ be a basis for X_0. For $a \leq k \leq n$, define $\psi_k : X_0 \to \mathbb{R}$ by $\psi_k \left(\sum_{i=1}^n \lambda_i e_i \right) = \lambda_k$. Since X_0 is finite dimensional and each ψ_k is clearly linear then each ψ_k is continuous by Exercise 13.26. By Theorem 14.7 each ψ_k has an extension ψ_k' to all of X. Since ψ_k is continuous then it is bounded by Theorem 13.1. ψ_k' is bounded as given by Theorem 14.7, so ψ_k' is continuous by Theorem 13.1. Since $\psi_k' : X \to \mathbb{R}$ and $\mathbb{R}$ is finite dimensional then by Exercise 13.2b, $\text{Ker}(\psi_k')$ is closed in X for each $1 \leq k \leq n$. So subspace $X_1 = \cap_{k=1}^n \text{Ker}(\psi_k')$ is closed in X.

The only element of X_0 in X_1 is 0. Also for $x \in X$ we have (similar to the proof of Lemma 14.1.A): $x = \left(\sum_{k=1}^n \psi_k'(x) e_k \right) + \left(x - \sum_{k=1}^n \psi_k'(x) e_k \right)$ where $\sum_{k=1}^n \psi_k'(x) e_k \in X_0$ and for each k

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Proof (continued).

$$\psi'_k\left(x - \sum_{i=1}^{n} \psi'_i(x)e_i\right) = \psi'_k(x) - \sum_{i=1}^{n} \psi'_i(x)\psi'_k(e_i) = \psi'_k(x) - \psi'_k(x)(1) = 0,$$

so $x - \sum_{k=1}^{n} \psi'_k(x)e_k \in \cap_{k=1}^{n} \mathrm{Ker}(\psi'_k)$. Therefore, $X = X_0 \oplus X_1$. □

Corollary 14.9

Corollary 14.9. Let X be a normed linear space. Then the natural embedding $J : X \to X^{**}$ is an isometry.

Proof. Recall that by definition $J(x)[\psi] = \psi(x)$ for all $x \in X$ and $\psi \in X^*$. Let $x \in X$. Recall that by the definition of the operator norm we have $|\psi(x)| \leq \|\psi\|\|x\|$ for all $\psi \in X^*$.

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Theorem 14.10

Theorem 14.10. Let X_0 be a subspace of the normed linear space X. Then a point $x \in X$ belongs to the closure of X_0 if and only if whenever a functional $\psi \in X^*$ vanishes on X_0, it also vanishes at x.

Proof. Let x be in the closure of X_0. Then there is a sequence $\{x_n\} \subset X_0$ such that $\{x_n\} \to x$ by Proposition 9.6. Since ψ is bounded then it is continuous (Theorem 14.1) and so $\lim_{n\to\infty} \psi(x_n) = \psi(x)$, or $\psi(x) = 0$.

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For the converse, let $x_0 \in X \setminus \overline{X_0}$. We need to show that there is $\psi \in X^*$ that vanishes on X_0 but $\psi(x_0) \neq 0$. Define $X = \overline{X}_0 \oplus [x_0]$ and $\psi : Z \to \mathbb{R}$ by $\psi(z + \lambda x_0) = \lambda$ for all $x \in \overline{X}_0$ and $\lambda \in \mathbb{R}$. Notice that $\psi(x_0) = 1$.

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Proof (continued). So for $x \in \overline{X}_0$ and $\lambda \in \mathbb{R}$, $\|x + \lambda x_0\| = |\lambda| \|(-1/\lambda)x - x_0\| \geq |\lambda| r$, or $|\lambda| \leq \|x + \lambda x_0\|/r$. So $|\psi(x + \lambda x_0)| = |\lambda| \leq (1/r)\|x + \lambda x_0\|$ and hence $\|\psi\| \leq 1/r$, so that ψ has a bounded extension to all of X.

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Corollary 14.11

Corollary 14.11. Let $\mathcal{S}$ be a subset of the normed linear space X. Then the linear span of $\mathcal{S}$ is dense in X if and only if whenever $\psi \in X^*$ vanishes on $\mathcal{S}$, then $\psi = 0$.

Proof. Let span$[\mathcal{S}]$ be dense in X. Then $\overline{\text{span}}[\mathcal{S}] = X$ and so every point of X is a limit point of span$[\mathcal{S}]$. If $\psi \in X^*$ vanishes on span$[\mathcal{S}]$ then ψ vanishes on X by Theorem 14.10 and $\psi = 0$.

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Suppose whenever $\psi \in X^*$ vanishes on span$[\mathcal{S}]$ then $\psi = 0$. ASSUME span$[\mathcal{S}]$ is not dense in X. Then there is some $x_0 \in X \setminus \overline{\text{span}}[\mathcal{S}]$.

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Theorem 14.12

Theorem 14.12. Let X be a normed linear space. Then every weakly convergent sequence in X is bounded. moreover, if $\{x_n\} \rightharpoonup x$ in X, then $\|x\| \leq \liminf \|x_n\|$.

Proof. Let $\{x_n\} \rightharpoonup x$ in X. Then, by the definition of weak convergence, $\lim_{n\to\infty} \psi(x_n) = \psi(x)$ for all $\psi \in X^*$. Recall that in Section 14.1 we defined $J(x) : X^* \to \mathbb{R}$ as $J(x)[\psi] = \psi(x)$ for all $\psi \in X^*$.

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So for given ψ, $\{J(x_n)[\psi]\} \to J(x)[\psi]$ (since $J(x) : X^* \to \mathbb{R}$ the convergence is in $\mathbb{R}$). Every convergent sequence of real numbers is bounded, so there is some $M_\psi \geq 0$ such that $|J(x_n)[\psi]| \leq M_\psi$ for all $n \in \mathbb{N}$.

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For the "moreover" part, we know by Theorem 14.7 that there is a functional $\psi \in X^*$ for which $\|\psi\| = 1$ and $\psi(x) = \|x\|$. Then $|\psi(x_n)| \leq \|\psi\|\|x_n\| = \|x_n\|$ for all $n \in \mathbb{N}$.

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