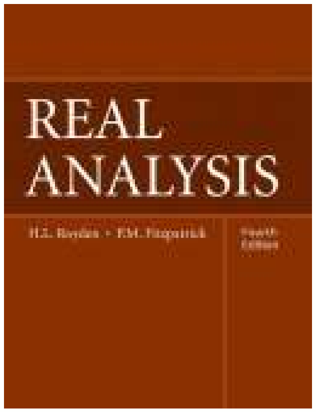
Real Analysis

Chapter 16. Continuous Linear Operators on Hilbert Spaces
16.1. The Inner Product and Orthogonality—Proofs of Theorems



Theorem. The Cauchy-Schwarz Inequality

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For any two vectors u and v in an inner product space H, $|\langle u, v\rangle| \le \|u\|\|v\|$ where $\|u\| = \sqrt{\langle u, u\rangle}$.

Proof. For any $t \in \mathbb{R}$ we have

$$0 \le \|u + tv\|^2 = \langle u + tv, u + tv\rangle = \langle u, u\rangle + 2t\langle u, v\rangle + t^2\langle v, v\rangle$$

$$= \|u\|^2 + 2t\langle u, v\rangle + t^2\|v\|^2.$$

Treating the right hand side as a quadratic in t and noticing that it cannot have distinct real roots (because if is nonnegative), we wee that the discriminant (i.e., the quantity "$b^2 - 4ac$") is not positive. That is, $(2\langle u, y\rangle)^2 - 4(\|v\|^2)(\|u\|^2) \le 0$ or $\langle u, v\rangle^2 \le \|v\|^2\|u\|^2$ or $|\langle u, v\rangle| \le \|u\|\|v\|$. □

Proposition 16.1

Proposition 16.1. For a vector h in an inner product space H, define $\|h\| = \sqrt{\langle h, h\rangle}$. Then $\|\cdot\|$ is a norm on H called the *norm induced* by the inner product $\langle\cdot,\cdot\rangle$.

Proof. First, for $h \in H$ and $\alpha \in \mathbb{R}$, we have $\|\alpha h\| = \sqrt{\langle \alpha h, \alpha h\rangle} = \sqrt{\alpha^2\langle h, h\rangle} = |\alpha|\|h\|$ so positive homogeneity holds. Next, $\|h\| = \sqrt{\langle h, h\rangle} \ge 0$ for all $h \in H$ and by definition $\|h\| = \sqrt{\langle h, h\rangle} > 0$ for $h \ne 0$. If $h = 0$, then be positive homogeneity $2\|h\| = \|2h\| = 2 \cdot 0\| = \|0\| = \|h\|$ and so $\|h\| = 0$ and nonnegativity holds. Finally, for $u, v \in H$ we have

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle = \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle\\ &= \|u\|^2 + 2\langle u, v\rangle + \|v\|^2\\ &\le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \text{ by the Cauchy-Schwarz Inequality}\\ &= (\|u\| + \|v\|)^2,\end{aligned}$$

so $\|u + v\| \le \|u\| + \|v\|$ and the triangle inequality holds. Therefore $\|\cdot\|$ is a norm on H. □

The Parallelogram Identity

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For any two vectors u, v in an inner product space H we have $\|u - v\|^2 + \|u + v\|^2 = 2\|u\|^2 + 2\|v\|^2$.

Proof. We have

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2$$

and

$$\|u-v\|^2 = \langle u-v, u-v\rangle = \langle u, u\rangle - 2\langle u, v\rangle + \langle v, v\rangle = \|u\|^2 - 2\langle u, v\rangle + \|v\|^2.$$

Adding the corresponding left and right sides of these equations yields the result. □

Proposition 16.2

Proposition 16.2. Let K be a nonempty closed convex subset of a Hilbert space H and let $j \in H \setminus K$. Then there is exactly one vector $h_* \in K$ that is closest to h_0 in the sense that $\|h_0 - h_*\| = \text{dist}(h_0, K) = \inf_{h \in K} \|h_0 - h\|$.

Proof. We prove the claim for $h_0 = 0$ (the general result then following be replacing K with $K - h_0$). Let $\{h_n\}$ be a sequence in K for which $\lim_{n\to\infty} \|h_n\| = \inf_{h\in K} \|h\|$. Then for any $m, n \in \mathbb{N}$ we have

$$\begin{aligned} \|h_n\|^2 + \|h_m\|^2 &= 2\left\|\frac{h_n + h_m}{2}\right\|^2 + 2\left\|\frac{h_n - h_m}{2}\right\|^2 \text{ by The Parallelogram} \\ &\quad \text{Identity with } u = (h_n + h_m)/2 \text{ and } v = (h_n - h_m)/2 \\ &\geq 2\inf_{h\in K} \|h\|^2 + 2\left\|\frac{h_n - h_m}{2}\right\| \text{ since } (h_n + h_m)/2 \in K \\ &\quad \text{because } K \text{ is convex.} \qquad (2) \end{aligned}$$

So $\|h_n\|^2 - \inf_{h\in K} \|h\|^2 + \|h_m\|^2 - \inf_{h\in K} \|h\| \geq \|h_n - h_m\|^2$ and since $\{h_n\} \to \inf_{h\in K} \|h\|$ then $\{h_n\}$ is a Cauchy sequence.

Proposition 16.2 (continued)

Proof (continued). Since H is complete then there is $h^* \in H$ such that $\{h_n\} \to h^*$. Since K is closed then it contains its limit points and so $h^* \in K$. By continuity of the norm (which follows from continuity of the induced metric; continuity of the metric follows from Exercise 9.14), $\|h^*\| = \inf_{h\in K} \|h\|$. This is a point in K closest to h_0. Suppose h^* is another vector in K that is closest to $h_0 = 0$. Then the sequence $\{h_n\}$ where $h_n = h_*$ or $h_n = h^*$ for all $n \in \mathbb{N}$ satisfies $\|h_n\| \to \inf_{h\in K} \|h\|$ (trivially) and so by equation (2) with $h_n = h^*$ and $m_m = h_*$ we have

$$0 \geq \|h^*\|^2 + \|h_*\|^2 - 2\inf_{h\in K} \|h\|^2 \geq 2\left\|\frac{h^* - h_*}{2}\right\|^2$$

and since $\|h^*\| = \|h_*\| = \inf_{h\in K} \|h\|$, we must have $h_* = h^*$ and so the closest element to $h_0 = 0$ is unique. □

Proposition 16.3

Proposition 16.3. Let V be a closed subspace of a Hilbert space H. Then H has the orthogonal direct sum decomposition $H = V \oplus V^\perp$.

Proof. Let $h_0 \in H \setminus V$. By Proposition 16.1 there is a unique $h^* \in V$ that is closest to h_0. Let $h \in V$. For $t \in \mathbb{R}$, since V is a linear space then $h^* - th \in V$ and therefore

$$\begin{aligned} \langle h_0 - h^*, h_0 - h^* \rangle &= \|h_0 - h^*\|^2 \\ &\leq \|h_0 - (h^* - th)\|^2 \text{ since } h^* \text{ is closest to } h_0 \\ &= \langle h_0 - (h^* - th), h_0 - (h^* - th) \rangle \\ &= \langle h_0 - h^*, h_0 - h^* \rangle + 2t\langle h_0 - h^*, h \rangle + t^2\langle h, h \rangle. \end{aligned}$$

Hence $0 \leq 2t\langle h_0 - h^*, h\rangle + t^2\|h\|^2$ for all $t \in \mathbb{R}$, or $(t\|h\|^2 + 2\langle h_0 - h^*, h\rangle)t \geq 0$. As a function of t, this is an opening upward parabola with intercepts at $t = 0$ and $t = -2\langle h_0 - h^*, h\rangle/\|h\|^2$ if $h \neq 0$. Such a parabola cannot have two intercepts since its graph is nonnegative.

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Proof. So it must be that $t = -2\langle h_0 - h^*, h\rangle/\|h\|^2 = 0$ also; that is, $\langle h_0 - h^*, h\rangle = 0$ (notice that if $h = 0$ then we still have $\langle h_0 - h^*, h\rangle = \langle h_0 - h^*, 0\rangle = 0$). So $h_0 - h^*$ is orthogonal to h and since h is an arbitrary vector in V then $h_0 - h^*$ is orthogonal to V. Next, $h_0 = h^* + (h_0 - h^*)$. So arbitrary $h_0 \in H \setminus V$ is a sum of an element of V (namely, h^*) and an element of $V^\perp$ (namely, $h_0 - h^*$). Of course any vector h_1 in V is the sum of an element of V (namely, h_1 itself) and an element of $V^\perp$ (namely, 0). So $H = V + V^\perp$. Now $V \cap B^\perp = \{0\}$ (the only element of V orthogonal to all elements of V is 0; in particular, the only vector orthogonal to itself is 0). Therefore $H = V \oplus V^\perp$. □

Proposition 16.5

Proposition 16.5. Let P be the orthogonal projection of a Hilbert space H onto a nontrivial closed subspace V of H. Then $\|P\| = 1$ and $\langle P(u), v\rangle = \langle u, P(v)\rangle$ for all $u, v \in H$.

Proof. Let $h \in H$. With the identity on H denoted as "Id" we have

$$\begin{aligned} \|u\|^2 &= \langle u,u\rangle = \langle P(u) + (\mathrm{Id} - P)(u), P(u) + (\mathrm{Id} - P)(u)\rangle \\ &= \|P(u)\|^2 + 2\langle P(u), (\mathrm{Id} - P)(u)\rangle + \|(\mathrm{Id} - P)(u)\|^2 \\ &= \|P(u)\|^2 + \|(\mathrm{Id} - P)(u)\|^2 \text{ since } P(u) \in V \text{ and} \\ &\quad u - P(u) \in V^\perp \text{ (by Theorem 16.3) so that} \\ &\quad \langle P(u), (\mathrm{Id} - P)(u)\rangle = 0 \\ &\geq \|P(u)\|^2, \end{aligned}$$

and hence $\|P(u)\| \leq \|u\|$. Therefore $\|P\| = \inf_{u \in H, n \neq 0} \frac{\|P(u)\|}{\|u\|} \leq 1$. Since $P(v) = v$ for all nonzero $v \in V$ (such v exists since V is nontrivial) and $\frac{\|P(v)\|}{\|v\|} = \frac{\|v\|}{\|v\|} = 1$, then $\|P\| = 1$.

Proposition 16.5 (continued)

Proposition 16.5. Let P be the orthogonal projection of a Hilbert space H onto a nontrivial closed subspace V of H. Then $\|P\| = 1$ and $\langle P(u), v\rangle = \langle u, P(v)\rangle$ for all $u, v \in H$.

Proof (continued). Now for $u, v \in H$ we have $\langle P(u), (\mathrm{Id} - P)(v)\rangle = 0$ since $P(u) \in V$ and $(\mathrm{Id} - P)(v) \in V^\perp$ and so $\langle P(u), v\rangle = \langle P(u), P(v)\rangle$. Also $\langle (\mathrm{Id} - P)(u), P(v)\rangle = 0$ since $P(v) \in V$ and $(\mathrm{Id} - P)(u) \in V^\perp$ and so $\langle u, P(v)\rangle = \langle P(u), P(v)\rangle$. Therefore $\langle P(u), P(v)\rangle = \langle P(u), v\rangle = \langle u, P(v)\rangle$, as claimed. □