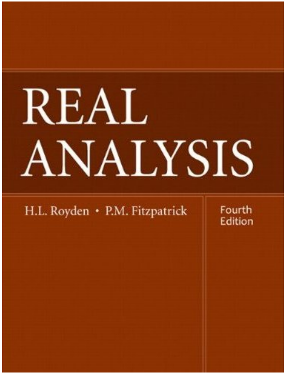
Real Analysis

Chapter 16. Continuous Linear Operators on Hilbert Spaces

16.1. The Inner Product and Orthogonality—Proofs of Theorems



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Treating the right hand side as a quadratic in t and noticing that it cannot have distinct real roots (because if is nonnegative), we wee that the discriminant (i.e., the quantity "$b^2 - 4ac$") is not positive. That is, $(2\langle u,y\rangle)^2 - 4(\|v\|^2)(\|u\|^2) \le 0$ or $\langle u,v\rangle^2 \le \|v\|^2\|u\|^2$ or $|\langle u,v\rangle| \le \|u\|\|v\|$. □

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Proof. First, for $h \in H$ and $\alpha \in \mathbb{R}$, we have $\|\alpha h\| = \sqrt{\langle \alpha h, \alpha h\rangle} = \sqrt{\alpha^2\langle h, h\rangle} = |\alpha|\|h\|$ so positive homogeneity holds.

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$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle = \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle \\ &= \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \text{ by the Cauchy-Schwarz Inequality} \\ &= (\|u\| + \|v\|)^2, \end{aligned}$$

so $\|u+v\| \leq \|u\| + \|v\|$ and the triangle inequality holds. Therefore $\|\cdot\|$ is a norm on H. □

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Proposition 16.2. Let K be a nonempty closed convex subset of a Hilbert space H and let $j \in H \setminus K$. Then there is exactly one vector $h_* \in K$ that is closest to h_0 in the sense that $\|h_0 - h_*\| = \text{dist}(h_0, K) = \inf_{h \in K} \|h_0 - h\|$.

Proof. We prove the claim for $h_0 = 0$ (the general result then following be replacing K with $K - h_0$). Let $\{h_n\}$ be a sequence in K for which $\lim_{n\to\infty} \|h_n\| = \inf_{h \in K} \|h\|$.

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$$\begin{aligned} \|h_n\|^2 + \|h_m\|^2 &= 2\left\|\frac{h_n + h_m}{2}\right\|^2 + 2\left\|\frac{h_n - h_m}{2}\right\|^2 \text{ by The Parallelogram} \\ &\quad \text{Identity with } u = (h_n + h_m)/2 \text{ and } v = (h_n - h_m)/2 \\ &\geq 2\inf_{h \in K} \|h\|^2 + 2\left\|\frac{h_n - h_m}{2}\right\| \text{ since } (h_n + h_m)/2 \in K \\ &\quad \text{because } K \text{ is convex.} \end{aligned} \tag{2}$$

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So $\|h_n\|^2 - \inf_{h \in K} \|h\|^2 + \|h_m\|^2 - \inf_{h \in K} \|h\| \geq \|h_n - h_m\|^2$ and since $\{h_n\} \to \inf_{h \in K} \|h\|$ then $\{h_n\}$ is a Cauchy sequence.

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Proof (continued). Since H is complete then there is $h^* \in H$ such that $\{h_n\} \to h^*$. Since K is closed then it contains its limit points and so $h^* \in K$. By continuity of the norm (which follows from continuity of the induced metric; continuity of the metric follows from Exercise 9.14), $\|h^*\| = \inf_{h \in K} \|h\|$. This is a point in K closest to h_0. Suppose h^* is another vector in K that is closest to $h_0 = 0$.

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$$0 \geq \|h^*\|^2 + \|h_*\|^2 - 2 \inf_{h \in K} \|h\|^2 \geq 2 \left\| \frac{h^* - h_*}{2} \right\|^2$$

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Hence $0 \leq 2t\langle h_0 - h^*, h\rangle + t^2\|h\|^2$ for all $t \in \mathbb{R}$, or $(t\|h\|^2 + 2\langle h_0 - h^*, h\rangle)t \geq 0$. As a function of t, this is an opening upward parabola with intercepts at $t = 0$ and $t = -2\langle h_0 - h^*, h\rangle / \|h\|^2$ if $h \neq 0$. Such a parabola cannot have two intercepts since its graph is nonnegative.

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Proof. So it must be that $t = -2\langle h_0 - h^*, h\rangle / \|h\|^2 = 0$ also; that is, $\langle h_0 - h^*, h\rangle = 0$ (notice that if $h = 0$ then we still have $\langle h_0 - h^*, h\rangle = \langle h_0 - h^*, 0\rangle = 0$). So $h_0 - h^*$ is orthogonal to h and since h is an arbitrary vector in V then $h_0 - h^*$ is orthogonal to V. Next, $h_0 = h^* + (h_0 - h^*)$. So arbitrary $h_0 \in H \setminus V$ is a sum of an element of V (namely, h^*) and an element of $V^\perp$ (namely, $h_0 - h^*$). Of course any vector h_1 in V is the sum of an element of V (namely, h_1 itself) and an element of $V^\perp$ (namely, 0). So $H = V + V^\perp$.

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Proposition 16.5

Proposition 16.5. Let P be the orthogonal projection of a Hilbert space H onto a nontrivial closed subspace V of H. Then $\|P\| = 1$ and $\langle P(u), v\rangle = \langle u, P(v)\rangle$ for all $u, v \in H$.

Proof. Let $h \in H$. With the identity on H denoted as "Id" we have

$$\begin{aligned} \|u\|^2 &= \langle u, u\rangle = \langle P(u) + (\mathrm{Id} - P)(u), P(u) + (\mathrm{Id} - P)(u)\rangle \\ &= \|P(u)\|^2 + 2\langle P(u), (\mathrm{Id} - P)(u)\rangle + \|(\mathrm{Id} - P)(u)\|^2 \\ &= \|P(u)\|^2 + \|(\mathrm{Id} - P)(u)\|^2 \text{ since } P(u) \in V \text{ and} \\ &\quad\; u - P(u) \in V^{\perp} \text{ (by Theorem 16.3) so that} \\ &\quad\; \langle P(u), (\mathrm{Id} - P)(u)\rangle = 0 \\ &\geq \|P(u)\|^2, \end{aligned}$$

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Proof (continued). Now for $u, v \in H$ we have $\langle P(u), (\mathrm{Id} - P)(v)\rangle = 0$ since $P(u) \in V$ and $(\mathrm{Id} - P)(v) \in V^{\perp}$ and so $\langle P(u), v\rangle = \langle P(u), P(v)\rangle$. Also $\langle (\mathrm{Id} - P)(u), P(v)\rangle = 0$ since $P(v) \in V$ and $(\mathrm{Id} - P)(u) \in V^{\perp}$ and so $\langle u, P(v)\rangle = \langle P(u), P(v)\rangle$. Therefore $\langle P(u), P(v)\rangle = \langle P(u), v\rangle = \langle u, P(v)\rangle$, as claimed. □

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