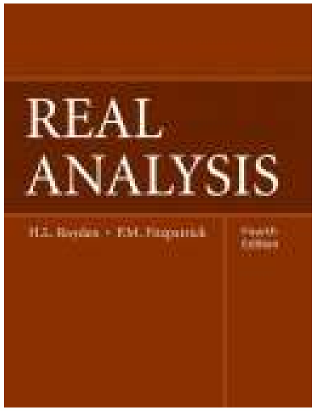
Real Analysis

Chapter 16. Continuous Linear Operators on Hilbert Spaces
16.2. The Dual Space and Weak Sequential Convergence—Proofs of Theorems



The Riesz-Fréchet Representation Theorem

The Riesz-Fréchet Representation Theorem.
Let H be a Hilbert space. Define the operator $T : H \to H^*$ (where H^* is the dual space of H, the linear space of all bounded linear functionals on H) by assigning to each $h \in H$ the linear functional $T(h) : H \to \mathbb{R}$ defined by $T(h)[u] = \langle h, u\rangle$ for all $h \in H$. Then T is a linear isometry of H onto H^*.

Proof. Let $h \in H$. Then for any $\alpha, \beta \in \mathbb{R}$ and $u, v \in H$ we have

$$T(h)[\alpha u+\beta v) = \langle h, \alpha u+\beta v\rangle = \alpha\langle h, u\rangle+\beta\langle h, v\rangle = \alpha T(h)[u]+\beta T(h)[v],$$

and so $T(h)$ is linear. By the Cauchy-Schwarz Inequality of Section 16.1, $|t(h)[u]| = |\langle h, u\rangle| \le \|h\|\|u\|$ or $|T(h)[u]|/\|u\| \le \|h\|$ and so $T(h)$ is bounded and $\|T(h)\| \le \|h\|$. But for $h \ne 0$ we have $T(h)[h/\|h\|] = T(h)[h]/\|h\| = \langle h, h\rangle/\|h\| = \|h\|^2/\|h\| = \|h\|$ and so $\|T(h)\| = \|h\|$. So $T : H \to H^*$ is an isometry (since for $h \in H$ and $T(h) \in H^*$ we have $\|h\| = \|T(h)\|$).

The Riesz-Fréchet Representation Theorem, continued 1

Proof (continued). T is linear since for $\alpha, \beta \in \mathbb{R}$ and $h_1, h_2 \in H$ we have

$$T(\alpha h_1+\beta h_2)[u] = \langle \alpha h_1+\beta h_2, u\rangle = \alpha\langle h_1, u\rangle_\beta\langle_2, u\rangle = \alpha T(h_1)[u]+\beta T(h_2)[u]$$

That is, T is a linear isometry.

To show $T : H \to H^*$ is onto, notice that $T(0)[u] = \langle 0, u\rangle = 0$ for all $u \in H$. So T maps $0 \in H$ to $0 \in H^*$. Next, let $\psi_0 \in H^*$ with $\psi_0 \ne 0$ (so $\psi_0 :\to \mathbb{R}$). Since ψ_0 is linear then it is continuous and since $\|0\|$ is closed in $\mathbb{R}$ then $\psi_0^{-1}(\{0\}) = \text{Ker}(\psi_0)$ is closed in H (see Exercise 11.25(i)) and since $\psi_0 \ne 0$ then $\text{Ker}(\psi_0)$ is a proper subspace of H. By Theorem 16.3, there is unit vector $h_* \in H$ that is orthogonal to $\text{Ker}(\psi_0)$. Define $h_0 = \psi_0(h_*)h_*$. Then for $h \in H$ we have that $h - (\psi_0(h)/\psi_0(h_*))h_* \in \text{Ker}(\psi_0)$ and so h_* is orthogonal to this vector:

$$\left\langle h - \frac{\psi_0(h)}{\psi_0(h_*)}h_*, h_*\right\rangle = 0.$$

The Riesz-Fréchet Representation Theorem, continued 2

The Riesz-Fréchet Representation Theorem.
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Proof (continued). That is,

$$\langle h, h_*\rangle - \frac{\psi_0(h)}{\psi_0(h_*)}\langle h_*, h_*\rangle = 0$$

or $\langle h, h_*\rangle - \psi_0(h)/\psi_0(h_*) = 0$ (since $\|h_*\| = 1$ by choice) or $\psi_0(h_*)\langle h, h_*\rangle - \psi_0(h) = 0$ or $\psi_0(h) = \langle h, \psi_0(h_*)h_*\rangle = \langle h, h_0\rangle = T(h_0)[h]$. That is, $\psi_0 \in H^*$ and $T(h_0) \in H^*$ are the same for all $h \in H$. Hence $T(h_0) = \psi_0$ and so T maps H onto H^*, as claimed. □

Theorem 16.6

Theorem 16.6. Every bounded sequence in a Hilbert space H has a weakly convergent subsequence.

Proof. Let $\{h_n\}_{n=1}^{\infty}$ be a bounded sequence in H. Define H_0 to be the closed linear span of $\{h_n\}$ (that is, the topological closure of the span of $\{h_n\}$; see page 254). Then H_0 is separable since the set of all linear combinations of elements of $\{h_n\}$ with rational coefficients is countable and dense in H_0. For each $n \in \mathbb{N}$, define $\psi_n \in H_0^*$ as $\psi_n(h) = \langle h_n, h \rangle$ for all $h \in H_0$. For $h \in H$ with $\|h\| = 1$ we have $|\psi_n(h)| = |\langle h_n, h \rangle| \le \|h_n\|\|h\|$ by the Cauchy-Schwarz Inequality, and so $\|\psi_n\| \le \|h_n\|$. Since $\{h_n\}$ is a bounded then $\{\psi_n\}$ is bounded. Then $\|\psi_n\}$ is a bounded sequence of bounded linear functionals on the separable linear space H_0. By Helley's Theorem (see page 283) there is a subsequence $\{\psi_{n_k}\}$ of $\{\psi_n\}$ that converges pointwise to some $\psi_0 \in H_0^*$. By the Riesz-Fréchet Representation Theorem, there is $h_0 \in H_0$ for which $\psi_0 = T(h_0)$ (that is, $\psi_0(h) = T(h_0)[h] = \langle h_0, h \rangle$ for all $h \in H_0$).

Theorem 16.6 (continued 1)

Theorem 16.6. Every bounded sequence in a Hilbert space H has a weakly convergent subsequence.

Proof (continued). Now the "pointwise convergence" of $\{\psi_{n_k}\}$ to ψ_0 means that for all points $h \in H_0$ we have

$$\lim_{k\to\infty} \psi_{n_k}(h) = \psi_0(h) \text{ of } \lim_{k\to\infty} \langle h_{n_k}, h \rangle = \langle h_0, h \rangle \text{ for all } h \in H_0. \qquad (*)$$

(This shows that $\{h_{n_k}\}$ converges weakly to h_0 in H_0; we must still show that $\{h_{n_k}\} \rightharpoonup h_0$ in H_0). Let P be the orthogonal projection mapping from H onto H_0 (so P projects $H = H_0 \oplus H_0^{\perp}$ onto H_0). For each $k \in \mathbb{N}$, since $(\mathrm{Id} - P)[H] = P(H)^{\perp} = H_0^{\perp}$, we have $\langle h_{n_k}, (\mathrm{Id} - P)[h] \rangle = 0$ for all $h \in H$, since $h_{n_k} \in H_0$ and $\langle h_0, (\mathrm{Id} - P)[h] \rangle = 0$ for all $h \in H$, since $h_0 \in H_0$. Next, for all $h \in H$ we have

$$\langle h_{n_k}, h \rangle = \langle h_{n_k}, (\mathrm{Id} - P)[h] \rangle = \langle h_{n_k}, (\mathrm{Id} - P)[h] \rangle + \langle h_{n_k}, P]h] \rangle = \langle h_{n_k}, P[h] \rangle$$

and similarly $\langle h_0, h \rangle = \langle h_0, P[h] \rangle$.

Theorem 16.6 (continued 2)

Theorem 16.6. Every bounded sequence in a Hilbert space H has a weakly convergent subsequence.

Proof (continued). Since $P[h] \in H_0$,

$$\begin{aligned} \lim_{k\to\infty} \langle h_{n_k}, h \rangle &= \lim_{k\to\infty} \langle h_{n_k}, P[h] \rangle \\ &= \langle h_0, P[h] \rangle \text{ by } (*) \\ &= \langle h_0, h \rangle \text{ for all } h \in H. \end{aligned}$$

Therefore, by definition, $\{h_{n_k}\}$ converges weakly to h_0 in H. □

The Banach-Saks Theorem

Theorem. The Banach-Saks Theorem.
Let $\{u_n\} \rightharpoonup u$ weakly in Hilbert space H. Then there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ for which

$$\lim_{k\to\infty} \frac{u_{n_1} + u_{n_2} + \cdots + u_{n_k}}{k} = u \text{ (strongly) in } H.$$

Proof. Replacing each u_n with $u_n - u$ we may suppose without loss of generality that $u = 0$. A weakly convergent sequence is bounded by Proposition 16.7, we may choose $M > 0$ such that $\|u_n\|^2 \le M$ for all $n \in \mathbb{N}$.

Define $n_1 = 1$. Since $\{u_n\} \rightharpoonup u = 0$ then, by definition, $\lim_{n\to\infty} \langle h, u_n \rangle = \langle h, 0 \rangle = 0$ for all $h \in H$ and so with $h = u_n = u_{n_1}$, there is some $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that $|\langle u_{n_1}, u_{n_2} \rangle| \le 1$. Then

$$\|u_{n_1} + u_{n_2}\|^2 = \langle u_{n_1} + u_{n_2}, u_{n_1} + u_{n_2} \rangle$$
$$= \|u_{n_1}\|^2 + 2\langle u_{n_1}, u_{n_2} \rangle + \|u_{n_2}\|^2 \le 2 + 2M \le 4 + 2M = (2 + M)2.$$

The Banach-Saks Theorem (continued 1)

Proof (continued). Suppose we have chosen natural numbers $n_1 < n_2 < \cdots < n_k$ such that $\|u_{n_1} + u_{n_2} + \cdots + u_{n_j}\| \le (2+M)j$ for $j = 1, 2, \ldots, k$. Since $\{u_n\| \rightharpoonup u = 0$ then $\lim_{n\to\infty}\langle h, u_n\rangle = \langle h, u\rangle = \langle h, 0\rangle = 0$, so with $h = u_{n_1} + u_{n_2} + \cdots + u_{n_k}$, there is some $n_{k+1} > n_k$ such that $|\langle u_{n_1} + u_{n_2} + \cdots + u_{n_k}, u_{n_{k+1}}\rangle| \le 1$. Then

$$\|u_{n_1} + u_{n_2} + \cdots + u_{n_k} + u_{n_{k+1}}\|^2$$
$$= \langle u_{n_1} + u_{n_2} + \cdots + u_{n_k} + u_{n_{k+1}}, u_{n_1} + u_{n_2} + \cdots + u_{n_k} + u_{n_{k+1}}\rangle$$
$$= \langle u_{n_1} + u_{n_2} + \cdots + u_{n_k}, u_{n_1} + u_{n_2} + \cdots + u_{n_k}\rangle + 2\langle u_{n_1} + u_{n_2} + \cdots + u_{n_k}, u_{n_{k+1}}\rangle$$
$$+ \langle u_{n_{k+1}}, u_{n_{k+1}}\rangle$$
$$= \|u_{n_1} + u_{n_2} + \cdots + u_{n_k}\|^2 + 2\langle u_{n_1} + u_{n_2} + \cdots + u_{n_k}, u_{n_{k+1}}\rangle + \|u_{n_{k+1}}\|^2$$
$$\le (2+M)k + 2 + M = (2+M)(k+1).$$

The Banach-Saks Theorem (continued 2)

Proof (continued). So by mathematical induction, for all $k \in \mathbb{N}$ we have $\|u_{n_1} + u_{n_2} + \cdots + u_{n_k}\|^2 \le (2+M)k$ or

$$\left\|\frac{u_{n_1} + u_{n_2} + \cdots + u_{n_k}}{k}\right\|^2 \le \frac{2+M}{k}.$$

Since M is fixed,

$$\lim_{k\to\infty}\left\|\frac{u_{n_1} + u_{n_2} + \cdots + u_{n_k}}{k}\right\| \le \lim_{k\to\infty}\sqrt{\frac{2+M}{k}} = 0.$$

Therefore,

$$\lim_{k\to\infty}\frac{u_{n_1} + u_{n_2} + \cdots + u_{n_k}}{k} = 0 = u,$$

and the claim holds. □

The Radon-Riesz Theorem

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Let $\{u_n\} \to u$ weakly (that is, $\{u_n\} \rightharpoonup u$) in the Hilbert space H. Then

$$\{u_n\} \to u \text{ strongly in } H \text{ if and only if } \lim_{n\to\infty}\|u_n\| = \|u\|.$$

Here, "strong convergence" means convergence with respect to the Hilbert space norm.

Proof. The norm on H is a continuous function from H to $\mathbb{R}$ by Exercise 13.4. So if $\{u_n\} \to u$ strongly in H then $\lim_{n\to\infty}\|u_n\| = \|\lim_{n\to\infty} u_n\| = \|u\|$. Conversely, if $\lim_{n\to\infty}\|u_n\| = \|u\|$ then

$$\|u_n - u\|^2 = \|u_n\|^2 - 2\langle u_n, u\rangle + \|u\|^2 \qquad (*)$$

for all $n \in \mathbb{N}$. With $\{u_n\} \to u$ we have (by definition) $\lim_{n\to\infty}\langle h, u_n\rangle = \langle h, u\rangle$ for all $h \in H$, so

$$\lim_{n\to\infty}\langle u_n, u\rangle = \lim_{n\to\infty}\langle u, u_n\rangle = \langle u, u\rangle = \|u\|^2.$$

The Radon-Riesz Theorem (continued)

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Here, "strong convergence" means convergence with respect to the Hilbert space norm.

Proof (continued). Therefore $\lim_{n\to\infty}\|u_n\|^2 - 2\langle u_n, u\rangle + \|u\|^2 = 0$ and so by $(*)$, $\lim_{n\to\infty}\|u_n - u\| = 0$. That is, $\{u_n\} \to u$ strongly in H. □