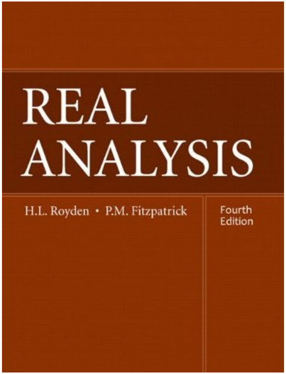
Real Analysis

Chapter 16. Continuous Linear Operators on Hilbert Spaces

16.2. The Dual Space and Weak Sequential Convergence—Proofs of Theorems



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and so $T(h)$ is linear. By the Cauchy-Schwarz Inequality of Section 16.1, $|t(h)[u]| = |\langle h, u \rangle| \le \|h\| \|u\|$ or $|T(h)[u]|/\|u\| \le \|h\|$ and so $T(h)$ is bounded and $\|T(h)\| \le \|h\|$. But for $h \neq 0$ we have $T(h)[h/\|h\|] = T(h)[h]/\|h\| = \langle h, h \rangle/\|h\| = \|h\|^2/\|h\| = \|h\|$ and so $\|T(h)\| = \|h\|$.

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Proof (continued). T is linear since for $\alpha, \beta \in \mathbb{R}$ and $h_1, h_2 \in H$ we have

$$T(\alpha h_1 + \beta h_2)[u] = \langle \alpha h_1 + \beta h_2, u \rangle = \alpha \langle h_1, u \rangle_\beta \langle_2, u \rangle = \alpha T(h_1)[u] + \beta T(h_2)[u]$$

That is, T is a linear isometry.

To show $T : H \to H^*$ is onto, notice that $T(0)[u] = \langle 0, u \rangle = 0$ for all $u \in H$. So T maps $0 \in H$ to $0 \in H^*$.

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Theorem 16.6

Theorem 16.6. Every bounded sequence in a Hilbert space H has a weakly convergent subsequence.

Proof. Let $\{h_n\}_{n=1}^{\infty}$ be a bounded sequence in H. Define H_0 to be the closed linear span of $\{h_n\}$ (that is, the topological closure of the span of $\{h_n\}$; see page 254). Then H_0 is separable since the set of all linear combinations of elements of $\{h_n\}$ with rational coefficients is countable and dense in H_0.

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(This shows that $\{h_{n_k}\}$ converges weakly to h_0 in H_0; we must still show that $\{h_{n_k}\} \rightharpoonup h_0$ in H_0). Let P be the orthogonal projection mapping from H onto H_0 (so P projects $H = H_0 \oplus H_0^\perp$ onto H_0). For each $k \in \mathbb{N}$, since $(\mathrm{Id} - P)[H] = P(H)^\perp = H_0^\perp$, we have $\langle h_{n_k}, (\mathrm{Id} - P)[h]\rangle = 0$ for all $h \in H$, since $h_{n_k} \in H_0$ and $\langle h_0, (\mathrm{Id} - P)[h]\rangle = 0$ for all $h \in H$, since $h_0 \in H_0$.

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and similarly $\langle h_0, h\rangle = \langle h_0, P[h]\rangle$.

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$$\begin{aligned} \lim_{k\to\infty} \langle h_{n_k}, h\rangle &= \lim_{k\to\infty} \langle h_{n_k}, P[h]\rangle \\ &= \langle h_0, P[h]\rangle \text{ by } (*) \\ &= \langle h_0, h\rangle \text{ for all } h \in H. \end{aligned}$$

Therefore, by definition, $\{h_{n_k}\}$ converges weakly to h_0 in H. □

The Banach-Saks Theorem

Theorem. The Banach-Saks Theorem.
Let $\{u_n\} \rightharpoonup u$ weakly in Hilbert space H. Then there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ for which

$$\lim_{k\to\infty} \frac{u_{n_1} + u_{n_2} + \cdots + u_{n_k}}{k} = u \text{ (strongly) in } H.$$

Proof. Replacing each u_n with $u_n - u$ we may suppose without loss of generality that $u = 0$. A weakly convergent sequence is bounded by Proposition 16.7, we may choose $M > 0$ such that $\|u_n\|^2 \leq M$ for all $n \in \mathbb{N}$.

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$$\|u_{n_1} + u_{n_2}\|^2 = \langle u_{n_1} + u_{n_2}, u_{n_1} + u_{n_2}\rangle$$

$$= \|u_{n_1}\|^2 + 2\langle u_{n_1}, u_{n_2}\rangle + \|u_{n_2}\|^2 \le 2 + 2M \le 4 + 2M = (2 + M)2.$$

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$$= \|u_{n_1}\|^2 + 2\langle u_{n_1}, u_{n_2}\rangle + \|u_{n_2}\|^2 \leq 2 + 2M \leq 4 + 2M = (2 + M)2.$$

The Banach-Saks Theorem (continued 1)

Proof (continued). Suppose we have chosen natural numbers $n_1 < n_2 < \cdots < n_k$ such that $\|u_{n_1} + u_{n_2} + \cdots + u_{n_j}\| \leq (2+M)j$ for $j = 1, 2, \ldots, k$. Since $\{u_n\| \rightharpoonup u = 0$ then $\lim_{n\to\infty}\langle h, u_n\rangle = \langle h, u\rangle = \langle h, 0\rangle = 0$, so with $h = u_{n_1} + u_{n_2} + \cdots + u_{n_k}$, there is some $n_{k+1} > n_k$ such that $|\langle u_{n_1} + u_{n_2} + \cdots + u_{n_k}, u_{n_{k+1}}\rangle| \leq 1$.

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$$
\begin{aligned}
&\|u_{n_1} + u_{n_2} + \cdots + u_{n_k} + u_{n_{k+1}}\|^2 \\
&= \langle u_{n_1} + u_{n_2} + \cdots + u_{n_k} + u_{n_{k+1}}, u_{n_1} + u_{n_2} + \cdots + u_{n_k} + u_{n_{k+1}}\rangle \\
&= \langle u_{n_1} + u_{n_2} + \cdots + u_{n_k}, u_{n_1} + u_{n_2} + \cdots + u_{n_k}\rangle + 2\langle u_{n_1} + u_{n_2} + \cdots + u_{n_k}, u_{n_{k+1}}\rangle \\
&\quad + \langle u_{n_{k+1}}, u_{n_{k+1}}\rangle \\
&= \|u_{n_1} + u_{n_2} + \cdots + u_{n_k}\|^2 + 2\langle u_{n_1} + u_{n_2} + \cdots + u_{n_k}, u_{n_{k+1}}\rangle + \|u_{n_{k+1}}\|^2 \\
&\le (2+M)k + 2 + M = (2+M)(k+1).
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The Banach-Saks Theorem (continued 2)

Proof (continued). So by mathematical induction, for all $k \in \mathbb{N}$ we have $\|u_{n_1} + u_{n_2} + \cdots + u_{n_k}\|^2 \leq (2 + M)k$ or

$$\left\| \frac{u_{n_1} + u_{n_2} + \cdots + u_{n_k}}{k} \right\|^2 \leq \frac{2 + M}{k}.$$

Since M is fixed,

$$\lim_{k\to\infty} \left\| \frac{u_{n_1} + u_{n_2} + \cdots + u_{n_k}}{k} \right\| \leq \lim_{k\to\infty} \sqrt{\frac{2 + M}{k}} = 0.$$

Therefore,

$$\lim_{k\to\infty} \frac{u_{n_1} + u_{n_2} + \cdots + u_{n_k}}{k} = 0 = u,$$

and the claim holds. □

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The Radon-Riesz Theorem

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Let $\{u_n\} \to u$ weakly (that is, $\{u_n\} \rightharpoonup u$) in the Hilbert space H. Then

$$\{u_n\} \to u \text{ strongly in } H \text{ if and only if } \lim_{n\to\infty} \|u_n\| = \|u\|.$$

Here, "strong convergence" means convergence with respect to the Hilbert space norm.

Proof. The norm on H is a continuous function from H to $\mathbb{R}$ by Exercise 13.4. So if $\{u_n\} \to u$ strongly in H then $\lim_{n\to\infty} \|u_n\| = \|\lim_{n\to\infty} u_n\| = \|u\|$. Conversely, if $\lim_{n\to\infty} \|u_n\| = \|u\|$ then

$$\|u_n - u\|^2 = \|u_n\|^2 - 2\langle u_n, u\rangle + \|u\|^2 \qquad (*)$$

for all $n \in \mathbb{N}$.

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