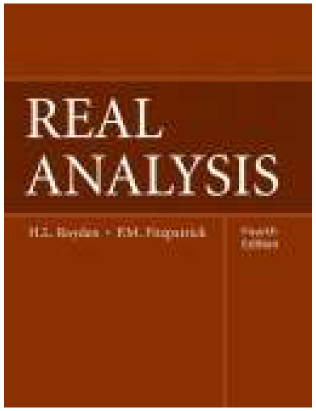
Real Analysis

Chapter 16. Continuous Linear Operators on Hilbert Spaces

16.3. Bessel's Inequality and Orthonormal Bases—Proofs of Theorems



The General Pythagorean Identity

Theorem. The General Pythagorean Identity.
If $u_1, u_2, \ldots, u_n$ are n orthonormal vectors in H, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ then

$$\|\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n\|^2 = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2.$$

Proof. We have

$$\|\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n\|^2 = \langle \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n,$$

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \rangle = \sum_{1 \le i,j \le n} \langle \alpha_i u_i, \alpha_j u_j \rangle$$

$$= \sum_{1 \le i,j \le n} \alpha_i \alpha_j \langle u_i, u_j \rangle = \sum_{1 \le i \le n} \alpha_i \alpha_i \langle u_i, u_i \rangle \text{ since } \langle u_i, u_j \rangle = 0 \text{ for } i \ne j$$

because $u_1, u_2, \ldots, u_n$ are orthogonal

The General Pythagorean Identity, continued

Theorem. The General Pythagorean Identity.
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$$\|\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n\|^2 = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2.$$

Proof (continued).

$$= \sum_{1 \le i \le n} \alpha_i^2 \text{ since } \langle u_i, u_i \rangle = \|u_i\|^2 = 1 \text{ because } u_1, u_2, \ldots, u_n \text{ are orthonormal}$$

$$= \sum_{1 \le i \le n} |\alpha_i|^2 - |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2.$$

□

Bessel's Inequality

Theorem. Bessel's Inequality.

For $\{\varphi_k\}$ an orthonormal sequence in H and $h \in H$, $\displaystyle\sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2 \le \|h\|^2$.

Proof. For fixed $n \in \mathbb{N}$ define $h_n = \sum_{k=1}^n \langle \varphi_k, h \rangle \varphi_k$. Then

$$\begin{aligned} 0 &\le \|h - h_n\|^2 = \|h\|^2 - 2\langle h, h_n \rangle + \|h_n\|^2 \\ &= \|h\|^2 - 2\left\langle h, \sum_{k=1}^n \langle \varphi_k, h \rangle \varphi_k \right\rangle + \|h_n\|^2 \\ &= \|h\|^2 - 2\sum_{k=1}^n \langle h, \varphi_k \rangle \langle \varphi_k, h \rangle + \sum_{k=1}^n \langle h, \varphi_k \rangle^2 \\ &\qquad \text{by the General Pythagorean Identity} \\ &= \|h\|^2 - \sum_{k=1}^n \langle h, \varphi_k \rangle^2. \end{aligned}$$

Bessel's Inequality (continued)

Theorem. Bessel's Inequality.
For $\{\varphi_k\}$ an orthonormal sequence in H and $h \in H$, $\displaystyle\sum_{k=1}^{\infty} \langle \varphi_k, h\rangle^2 \le \|h\|^2$.

Proof (continued) . Therefore

$$\sum_{k=1}^{n} \langle h, \varphi_k\rangle^2 \le \|h\|^2.$$

Since n is arbitrary,

$$\sum_{k=1}^{\infty} \langle h, \varphi_k\rangle^2 \le \|h\|^2,$$

as claimed. □

Proposition 16.9

Proposition 16.9. Let $\{\varphi_k\}$ be an orthonormal sequence in a Hilbert space H and let $h \in H$. Then the series $\sum_{k=1}^{\infty} \langle \varphi_k, h\rangle \varphi_k$ converges strongly in H and the vector $h - \sum_{k=1}^{\infty} \langle \varphi_k, h\rangle \varphi_k$ is orthogonal to each φ_k.

Proof. For each $n \in \mathbb{N}$, define $h_n = \sum_{k=1}^{n} \langle \varphi_k, h\rangle \varphi_k$. By the General Pythagorean Identity, for each pair of natural number n and k,

$$\|h_{n+k} - h_n\|^2 = \left\| \sum_{i=n+1}^{n+k} \langle \varphi_i, h\rangle \varphi_i \right\|^2 = \sum_{i=n+1}^{n+k} \langle \varphi_i, h\rangle^2.$$

By Bessel's Inequality, $\sum_{i=1}^{\infty} \langle \varphi_i, h\rangle^2 \le \|h\|^2$ so for $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that for all $m, n \ge N_\varepsilon$ (say $m \ge n$) we have

$$\|h_m - n_n\|^2 = \sum_{i=n_1}^{m} \langle \varphi_1, h\rangle^2 \le \sum_{i=n+1}^{\infty} \langle \varphi_i, h\rangle^2 < \varepsilon$$

(since the tail of a convergent sequence of real numbers must get small).

Proposition 16.9 (continued 1)

Proof. Therefore $\{h_n\}$ is a Cauchy sequence in H. Since H is complete then $\{h_n\} = \sum_{k=1}^{n} \langle \varphi_k, h\rangle \varphi_k \to h_* \in H$. That is, $\sum_{k=1}^{n} \langle \varphi_k, h\rangle \varphi_k$ converges strongly in H.
Fix $m \in \mathbb{N}$. If $n > m$, then

$$\begin{aligned}
\langle h - h_n, \varphi_m\rangle &= \left\langle h - \sum_{k=1}^{n} \langle \varphi_k, h\rangle \varphi_k, \varphi_m \right\rangle = \langle h, \varphi_m\rangle - \sum_{k=1}^{n} \langle \varphi_k, h\rangle \langle \varphi_k, \varphi_m\rangle \\
&= \langle h, \varphi_m\rangle - \langle \varphi_m, h\rangle\langle \varphi_m, \varphi_m\rangle \text{ since } \langle \varphi_k, \varphi_m\rangle = 0 \text{ for } k \ne m \\
&\qquad \text{because } \{\varphi_k\} \text{ is an orthogonal set} \\
&= \langle h, \varphi_m\rangle - \langle \varphi_m, h\rangle \text{ since } \|\varphi_m\|^2 = \langle \varphi_m, \varphi_m\rangle = 1 \\
&= 0.
\end{aligned}$$

By the continuity of the inner product (that is, for $\{u_n\} \to u$ and $v \in H$, $\lim_{n\to\infty} \langle u_n, v\rangle = \langle u, v\rangle$; this follows from Proposition 16.7) we have

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Proposition 16.9 (continued 2)

Proposition 16.9. Let $\{\varphi_k\}$ be an orthonormal sequence in a Hilbert space H and let $h \in H$. Then the series $\sum_{k=1}^{\infty} \langle \varphi_k, h\rangle \varphi_k$ converges strongly in H and the vector $h - \sum_{k=1}^{\infty} \langle \varphi_k, h\rangle \varphi_k$ is orthogonal to each φ_k.

Proof (continued).

$$h - h_* = h - \sum_{k=1}^{\infty} \langle \varphi_k, h\rangle \varphi_k = h - \lim_{n\to\infty} \left(\sum_{k=1}^{n} \langle \varphi_k, h\rangle \varphi_k \right) = h - \lim_{n\to\infty} h_n$$

and so

$$\langle h - h_*, \varphi_k = \left\langle h - \lim_{n\to\infty} h_n, \varphi_m \right\rangle = \lim_{n\to\infty} \langle h - h_n, \varphi_m\rangle = 0.$$

Since $m \in \mathbb{N}$ is arbitrary, $h - h_*$ is orthogonal to φ_m for all $m \in \mathbb{N}$. □

Lemma 16.3.A

Lemma 16.3.A. Let $\{\varphi_k\}$ be an orthonormal sequence in Hilbert space H. Then $\{\varphi_k\}$ is complete if and only if the closed linear span of $\{\varphi_k\}$ is H.

Proof. Suppose $S = \{\varphi_k\}$ is complete. Then the only vector that is orthogonal to every element of S is 0; that is, $S^\perp = \{0\}$. So by Corollary 16.4, the linear space of S is all of H.

Suppose the closed linear span of S is H. Then by Corollary 16.4, $S^\perp = \{0\}$. That is, the only element of H orthogonal to every element of $S = \{\varphi_k\}$ is 0. So, by the definition of "complete," $S = \{\varphi_k\}$ is complete. □

Proposition 16.10

Proposition 16.10. An orthonormal sequence $\{\varphi_k\}$ is a Hilbert space H is complete if and only if it is an orthonormal basis.

Proof. First, assume $\{\varphi_k\}$ is complete. Since $\{\varphi_k\}$ is an orthonormal sequence. Then by Proposition 16.9, for any $h \in H$, $h - \sum_{k=1}^\infty \langle \varphi_k, h\rangle \varphi_k$ is orthogonal to wach φ_k. Since $\{\varphi_k\}$ is complete then, by definition, the only vector orthogonal to every φ_k is 0. So for all $h \in H$ we have $h = \sum_{k=1}^\infty \langle \varphi_k, h\rangle \varphi_k$. Therefore $\{\varphi_k\}$ is an orthonormal basis for H.

Conversely, suppose $\{\varphi_k\}$ is an orthonormal basis for H. Then if $h \in H$ is orthogonal to all φ_k then $h = \sum_{k=1}^\infty \langle \varphi_k, h\rangle \varphi_k = \sum_{k=1}^\infty 0\varphi_k = 0$. So the only vector $h \in H$ that is orthogonal to every φ_k is $h = 0$. That is, $\{\varphi_k\}$ is (by definition) complete. □

Theorem 16.11

Theorem 16.11. Every infinite dimensional separable Hilbert space has an orthonormal basis.

Proof. Let $\mathcal{F}$ be the collection of subsets of H that are orthonormal. Order $\mathcal{F}$ by inclusion. For every linearly ordered subcollection of $\mathcal{F}$, the union of the sets in the subcollection is an upper bound for the subcollection. By Zorn's Lemma, there is a maximal subset S_0 of $\mathcal{F}$. Since H is separable and S_0 is orthonormal, then by Exercise 16.19, S_0 is countable. Let $\{\varphi_k\}_{k=1}^\infty$ be an enumeration of S_0. If $h \in H$, $h \neq 0$, then by Proposition 16.9, $h - \sum_{k=1}^\infty \langle \varphi_k, h\rangle \varphi_k$ is orthogonal to each φ_k. Therefore $h - \sum_{k=1}^\infty \langle \varphi_k, h\rangle \varphi_k$ could be added to S_0 thus violating its maximality. So $h = \sum_{k=1}^\infty \langle \varphi_k, h\rangle \varphi_k$ and so $\{\varphi_k\}$ is an orthonormal basis for H. □