Chapter 2. Lebesgue Measure
2.5. Countable Additivity, Continuity, and the Borel-Cantelli Lemma—Proofs of Theorems
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Theorem 2.15. Measure is Continuous.

Lebesgue measure satisfies:

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets (i.e., $A_k \subset A_{k+1}$), then
$$m(\bigcup_{k=1}^{\infty} A_k) = m(\lim_{k \to \infty} A_k) = \lim_{k \to \infty} m(A_k).$$

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets (i.e., $B_k \supset B_{k+1}$) and $m(B_1) < \infty$, then
$$m(\bigcap_{k=1}^{\infty} B_k) = m(\lim_{k \to \infty} B_k) = \lim_{k \to \infty} m(B_k).$$

Proof of (i). If $m(A_{k_0}) = \infty$ for some $k_0$, then the result holds trivially. So suppose, without loss of generality, that $m(A_k) < \infty$ for all $k$. 
Theorem 2.15. Measure is Continuous.

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(i) If \( \{A_k\}_{k=1}^{\infty} \) is an ascending collection of measurable sets (i.e., \( A_k \subset A_{k+1} \)), then
\[
m(\bigcup_{k=1}^{\infty} A_k) = m(\lim_{k \to \infty} A_k) = \lim_{k \to \infty} m(A_k).
\]

(ii) If \( \{B_k\}_{k=1}^{\infty} \) is a descending collection of measurable sets (i.e., \( B_k \supset B_{k+1} \)) and \( m(B_1) < \infty \), then
\[
m(\bigcap_{k=1}^{\infty} B_k) = m(\lim_{k \to \infty} B_k) = \lim_{k \to \infty} m(B_k).
\]

Proof of (i). If \( m(A_{k_0}) = \infty \) for some \( k_0 \), then the result holds trivially. So suppose, without loss of generality, that \( m(A_k) < \infty \) for all \( k \). Define \( A_0 = \emptyset \) and \( C_k = A_k \setminus A_{k-1} \) for \( k \geq 1 \). Since \( \{A_k\} \) is ascending, the \( C_k \)'s are disjoint and \( \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} C_k \). Since \( m \) is countably additive by Proposition 2.6,
\[
m(\bigcup_{k=1}^{\infty} A_k) = m(\bigcup_{k=1}^{\infty} C_k) = \sum_{k=1}^{\infty} m(C_k) = \sum_{k=1}^{\infty} m(A_k \setminus A_{k-1}).
\]
Theorem 2.15. Measure is Continuous.

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(i) If \( \{A_k\}_{k=1}^{\infty} \) is an ascending collection of measurable sets (i.e., \( A_k \subset A_{k+1} \)), then
\[
m(\bigcup_{k=1}^{\infty} A_k) = m(\lim_{k \to \infty} A_k) = \lim_{k \to \infty} m(A_k).
\]

(ii) If \( \{B_k\}_{k=1}^{\infty} \) is a descending collection of measurable sets (i.e., \( B_k \supset B_{k+1} \)) and \( m(B_1) < \infty \), then
\[
m(\bigcap_{k=1}^{\infty} B_k) = m(\lim_{k \to \infty} B_k) = \lim_{k \to \infty} m(B_k).
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\]
Theorem 2.15 (continued 1)

Theorem 2.15. Measure is Continuous.
Lebesgue measure satisfies:

1. If \( \{A_k\}_{k=1}^{\infty} \) is an ascending collection of measurable sets (i.e., \( A_k \subset A_{k+1} \)), then
   \[
   m(\bigcup_{k=1}^{\infty} A_k) = m(\lim_{k \to \infty} A_k) = \lim_{k \to \infty} m(A_k).
   \]

Proof (continued). By the Excision Property of measure (Lemma 2.4.A),

\[
m(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} m(A_k \setminus A_{k-1}) = \sum_{k=1}^{\infty} [m(A_k) - m(A_{k-1})]
\]

\[
= \lim_{n \to \infty} \left( \sum_{k=1}^{n} [m(A_k) - m(A_{k-1})] \right) = \lim_{n \to \infty} [m(A_n) - m(A_0)] = \lim_{n \to \infty} m(A_n),
\]

since \( m(A_0) = m(\emptyset) = 0 \). Therefore

\[
m(\bigcup_{k=1}^{\infty} A_k) = m(\lim_{k \to \infty} A_k) = \lim_{k \to \infty} m(A_k), \text{ as claimed.}
\]
Theorem 2.15. Measure is Continuous.

Lebesgue measure satisfies:

(ii) If \( \{ B_k \}_{k=1}^{\infty} \) is a descending collection of measurable sets (i.e., \( B_k \supset B_{k+1} \)) and \( m(B_1) < \infty \), then

\[
m(\cap_{k=1}^{\infty} B_k) = m(\lim_{k \to \infty} B_k) = \lim_{k \to \infty} m(B_k).
\]

**Proof of (ii).** Define \( D_k = B_1 \setminus B_k \) for \( k \in \mathbb{N} \). Since \( \{ B_k \}_{k=1}^{\infty} \) is a descending sequence of sets, then \( \{ D_k \}_{k=1}^{\infty} \) is an ascending sequence of sets. Applying (i) to \( \{ D_k \}_{k=1}^{\infty} \) gives

\[
m(\cup_{k=1}^{\infty} D_k) = \lim_{k \to \infty} m(D_k). \quad (*)
\]

By De Morgan’s Laws (Theorem 0.1, applied to relative complements)

\[
\cup_{k=1}^{\infty} D_k = \cup_{k=1}^{\infty} (B_1 \setminus B_k) = \cup_{k=1}^{\infty} (B_1 \cap B_k^c) = B_1 \setminus \cap_{k=1}^{\infty} B_k. \quad (**)
\]
Theorem 2.15. Measure is Continuous.

Lebesgue measure satisfies:

(ii) If \( \{B_k\}_{k=1}^{\infty} \) is a descending collection of measurable sets (i.e., \( B_k \supset B_{k+1} \)) and \( m(B_1) < \infty \), then

\[
m(\cap_{k=1}^{\infty} B_k) = m(\lim_{k \to \infty} B_k) = \lim_{k \to \infty} m(B_k).
\]

Proof of (ii). Define \( D_k = B_1 \setminus B_k \) for \( k \in \mathbb{N} \). Since \( \{B_k\}_{k=1}^{\infty} \) is a descending sequence of sets, then \( \{D_k\}_{k=1}^{\infty} \) is an ascending sequence of sets. Applying (i) to \( \{D_k\}_{k=1}^{\infty} \) gives

\[
m(\bigcup_{k=1}^{\infty} D_k) = \lim_{k \to \infty} m(D_k).
\]

By De Morgan’s Laws (Theorem 0.1, applied to relative complements)

\[
\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} (B_1 \setminus B_k) = \bigcup_{k=1}^{\infty} (B_1 \cap B_k^c) = B_1 \setminus \bigcap_{k=1}^{\infty} B_k.
\]
Proof (continued). Next, by the Excision Property (Lemma 2.4.A), since \( m(B_k) < \infty \) and \( B_k \subset B_1 \), we have

\[
m(D_k) = m(B_1 \setminus B_k) = m(B_1) - m(B_k)
\]

for all \( k \in \mathbb{N} \). So

\[
m(\bigcup_{k=1}^{\infty} D_k) = m(B_1 \setminus \bigcap_{k=1}^{\infty} B_k) \text{ by (**)}
= m(B_1) - m(\bigcap_{k=1}^{\infty} B_k) \text{ by the Excision Property}
= \lim_{k \to \infty} m(D_k) \text{ by (*)}
= \lim_{k \to \infty} (m(B_1) - m(B_k)) \text{ by the definition of } D_k
= m(B_1) - \lim_{k \to \infty} m(B_k).
\]

Hence, since \( m(B_1) < \infty \), \( m(\bigcap_{k=1}^{\infty} B_k) = \lim_{k \to \infty} m(B_k) \), as claimed. \qed
Proof (continued). Next, by the Excision Property (Lemma 2.4.A), since $m(B_k) < \infty$ and $B_k \subset B_1$, we have

$$m(D_k) = m(B_1 \setminus B_k) = m(B_1) - m(B_k)$$

for all $k \in \mathbb{N}$. So

$$m(\bigcup_{k=1}^{\infty} D_k) = m(B_1 \setminus \bigcap_{k=1}^{\infty} B_k) \text{ by (**)}$$

$$= m(B_1) - m(\bigcap_{k=1}^{\infty} B_k) \text{ by the Excision Property}$$

$$= \lim_{k \to \infty} m(D_k) \text{ by (*)}$$

$$= \lim_{k \to \infty} (m(B_1) - m(B_k)) \text{ by the definition of } D_k$$

$$= m(B_1) - \lim_{k \to \infty} m(B_k).$$

Hence, since $m(B_1) < \infty$, $m(\bigcap_{k=1}^{\infty} B_k) = \lim_{k \to \infty} m(B_k)$, as claimed. \qed
The Borel-Cantelli Lemma

Let \( \{ E_k \}_{k=1}^{\infty} \) be a countable collection of measurable sets for which \( \sum_{k=1}^{\infty} m(E_k) < \infty \). Then almost all \( x \in \mathbb{R} \) belong to at most finitely many of the \( E_k \)'s.

**Proof.** By countable subadditivity \( m(\bigcup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} m(E_k) < \infty \). So

\[
m(\bigcap_{n=1}^{\infty} [\bigcup_{k=n}^{\infty} E_k]) = \lim_{n \to \infty} m(\bigcup_{k=n}^{\infty} E_k) \quad \text{by Theorem 2.15(ii)}
\]

\[
\leq \lim_{n \to \infty} \sum_{k=n}^{\infty} m(E_k) \quad \text{as above}
\]

\[
= 0 \quad \text{since} \quad \sum_{n=1}^{\infty} m(E_k) < \infty.
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\[
m(\bigcap_{n=1}^{\infty} \left[ \bigcup_{k=n}^{\infty} E_k \right]) = \lim_{n \to \infty} m(\bigcup_{k=n}^{\infty} E_k) \text{ by Theorem 2.15(ii)}
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= 0 \text{ since } \sum_{n=1}^{\infty} m(E_k) < \infty.
\]

Now \( \bigcap_{n=1}^{\infty} \left[ \bigcup_{k=n}^{\infty} E_k \right] \) is the set of all points which are in infinitely many \( E_k \)'s. Since the measure of this set is zero, almost all real numbers belong to finitely many \( E_k \)'s, as claimed.

\( \square \)
The Borel-Cantelli Lemma

Let \( \{E_k\}_{k=1}^\infty \) be a countable collection of measurable sets for which \( \sum_{k=1}^\infty m(E_k) < \infty \). Then almost all \( x \in \mathbb{R} \) belong to at most finitely many of the \( E_k \)'s.

**Proof.** By countable subadditivity \( m(\bigcup_{k=n}^\infty E_k) \leq \sum_{k=n}^\infty m(E_k) < \infty \). So

\[
m(\bigcap_{n=1}^\infty [\bigcup_{k=n}^\infty E_k]) = \lim_{n \to \infty} m(\bigcup_{k=n}^\infty E_k) \text{ by Theorem 2.15(ii)}
\]

\[
\leq \lim_{n \to \infty} \sum_{k=n}^\infty m(E_k) \text{ as above}
\]

\[
= 0 \text{ since } \sum_{n=1}^\infty m(E_k) < \infty.
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Now \( \bigcap_{n=1}^\infty [\bigcup_{k=n}^\infty E_k] \) is the set of all points which are in infinitely many \( E_k \)'s. Since the measure of this set is zero, almost all real numbers belong to finitely many \( E_k \)'s, as claimed.