Real Analysis

Chapter 2. Lebesgue Measure

2.6. Nonmeasurable Sets (3rd Ed.)—Proofs of Theorems

**Lemma 2.6.A**

**Lemma 2.6.A.** Let $E \subseteq [0, 1]$ and $E \in \mathcal{M}$. Then for all $y \in [0, 1]$, $E \uparrow y$ is measurable and $m(E \uparrow y) = m(E)$.

**Proof.** Define $E_1 = E \cap [0, 1 - y)$ and $E_2 = E \cap [1 - y, 1)$. Then $E_1 \cap E_2 = \emptyset$, $E = E_1 \cup E_2$, and $E_1, E_2 \in \mathcal{M}$. So $m(E) = m(E_1) + m(E_2)$ by countable additivity (Proposition 2.13). Now $E_1 \uparrow y = E_1 + y$ and so $E_1 \uparrow y \in \mathcal{M}$ and $m(E_1 \uparrow y) = m(E_1)$ since $m$ is translation invariant (Proposition 2.2). Also, $E_2 \uparrow y = (E_2 + y) - 1 = E_2 + (y - 1)$ and so $E_2 \uparrow y \in \mathcal{M}$ and $m(E_2 \uparrow y) = m(E_2)$. Next, $E \uparrow y = (E_1 \uparrow y) \cup (E_2 \uparrow y)$ and $(E_1 \uparrow y) \cap (E_2 \uparrow y) = \emptyset$, so $E \uparrow y \in \mathcal{M}$ and so by countable additivity (Proposition 2.13):

$$m(E \uparrow y) = m(E_1 \uparrow y) + m(E_2 \uparrow y) = m(E_1) + m(E_2) = m(E).$$

**Theorem 2.6.B**

**Theorem 2.6.B.** Set $P$ is not measurable.

**Proof.** First, we establish some set theoretic results. Let $\{r_i\}_{i=0}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0, 1)$ where $r_0 = 0$. Define $P_i = P \uparrow r_i$. Then $P_0 = P$.

If $x \in P_i \cap P_j$, then $x = p_i + r_i = p_j + r_j$ where $p_i, p_j \in P$. But then $p_i + (-r_j) = r_j + (-r_i) \in \mathbb{Q}$ and so $p_i \sim p_j$. So $p_i$ and $p_j$ are from the same equivalence class under $\sim$ and since $P$ contains only one representative from each equivalence class, then $p_i = p_j$ and $P_i = P_j$. Therefore $P_i \cap P_j = \emptyset$ if $i \neq j$ and so the $P_i$'s are disjoint and $\bigcup_{i=1}^{\infty} P_i \subseteq [0, 1)$.

Let $x \in [0, 1)$. Then $x$ is in some equivalence class $E_x$. Let $p_x \in P$ be the representative of class $E_x$ (i.e., $f(E_x) = p_x$ for choice function $f$). Then $p_x + q = x$ for some $q \in \mathbb{Q} \cap [0, 1)$ and so $x \in \bigcup_{i=1}^{\infty} (P \uparrow r_i) = \bigcup_{i=1}^{\infty} P_i$. Hence, since $x$ is an arbitrary element of $[0, 1)$ then $[0, 1) \subseteq \bigcup_{i=1}^{\infty} P_i$.

Therefore, $\bigcup_{i=1}^{\infty} P_i = [0, 1)$.

**Theorem 2.6.B (continued)**

**Theorem 2.6.B.** Set $P$ is not measurable.

**Proof (continued).** Assume $P$ is measurable. Then by Lemma 2.6.A, each $P_i$ is measurable and $m(P_i) = m(P)$. Hence

$$1 = m([0, 1)) \quad \text{by Propositions 2.1 and 2.8}$$

$$= m(\bigcup_{i=1}^{\infty} P_i) \quad \text{since $[0, 1) = \bigcup_{i=1}^{\infty} P_i$}$$

$$= \sum_{i=1}^{\infty} m(P_i) \quad \text{by countable additivity (Proposition 2.13)}$$

$$= \sum_{i=1}^{\infty} m(P) \quad \text{since $m(P) = m(P_i)$ for all $i \in \mathbb{N} \cup \{0\}$}$$

$$= \left\{ \begin{array}{ll}
0 & \text{if } m(P) = 0 \\
\infty & \text{if } m(P) > 0,
\end{array} \right.$$

a CONTRADICTION. Therefore the assumption that $P$ is measurable is false and so $P$ is not measurable.
Theorem 2.18.

There are disjoint sets of real numbers $A$ and $B$ for which

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

**Proof.** ASSUME $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets $A$ and $B$. Then for any $A, E \subseteq \mathbb{R}$ we have

$$m^*(A) = m^*((A \cap E) \cup (A \cap E^c)) = m^*(A \cap E) + m^*(A \cap E^c)$$

and so every $E \subseteq \mathbb{R}$ is measurable, a CONTRADICTION to Corollary 2.6.C. So for some disjoint $A, B \subseteq \mathbb{R}$ we have

$m^*(A \cup B) \neq m^*(A) + m^*(B)$. By subadditivity (Proposition 2.3) $m^*(A \cup B) \leq m^*(A) + m^*(B)$, so it must be that for some disjoint $A, B \subseteq \mathbb{R}$ we have $m^*(A \cup B) < m^*(A) + m^*(B)$. \qed