Chapter 3. Lebesgue Measurable Functions
3.3. Littlewood’s Three Principles, Egoroff’s Theorem, and Lusin’s
Theorem—Proofs of Theorems
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Lemma 3.10. Assume $E$ has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on $E$ that converges pointwise on $E$ to the real-valued function $f$. Then for each $\eta > 0$ and $\delta > 0$, there is a measurable subset $A$ of $E$ and an index $N$ for which

$$|f_n - f| < \eta \text{ on } A \text{ for all } n \geq N \text{ and } m(E \setminus A) < \delta.$$ 

Proof. For each $k$, the function $|f - f_k|$ is well-defined (since $f$ is real-valued then we do not have $\infty - \infty$ concerns, even though $f_k$ might be extended real valued) and measurable by Theorem 3.6 and Proposition 3.9. So $\{x \in E \mid |f(x) - f_k(x)| < \eta\}$ is measurable for all $\eta \in \mathbb{R}$. 
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Proof (continued). By continuity of measure (Theorem 2.15) $m(E) = \lim m(E_n)$. Since $m(E) < \infty$, we may choose $N \in \mathbb{N}$ such that $m(E_N) > m(E) - \delta$. Define $A = E_N$. Then by the Excision Property, $m(E \setminus A) = m(E) - m(A) = m(E) - m(E_N) < \delta$. \qed
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Egoroff’s Theorem

**Egoroff’s Theorem.** Assume $E$ has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on $E$ that converges pointwise on $E$ to the real-valued function $f$. Then for each $\varepsilon > 0$, there is a closed set $F$ contained in $E$ for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \setminus F) < \varepsilon.$$

**Proof.** Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Let $\delta = \varepsilon/2^{n+1}$ and $\eta = 1/n$. 


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Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Let $\delta = \varepsilon/2^{n+1}$ and $\eta = 1/n$. Then by Lemma 3.10 (this is where finite measure is used) there exists measurable $A_n \subset E$ and $N(n) \in \mathbb{N}$ such that $|f_k - f| < \eta = 1/n$ on $A_n$ for all $k \geq N(n)$ and $m(E \setminus A_n) < \delta = \varepsilon/2^{n+1}$. Define $A = \cap_{n=1}^{\infty} A_n$. 
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\[
m(E \setminus A) = m(E \setminus (\bigcap_{n=1}^{\infty} A_n)) = m(\bigcup_{n=1}^{\infty} (E \setminus A_n)) \text{ by DeMorgan’s Laws}
\leq \sum_{n=1}^{\infty} m(E \setminus A_n) \text{ by countable subadditivity}
\]

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$$m(E \setminus A) = m(E \setminus (\cap_{n=1}^{\infty} A_n))$$

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Proof (continued).

\[ m(E \setminus A) \leq \sum_{n=1}^{\infty} m(E \setminus A_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}. \]

We now show that \( \{f_n\} \rightarrow f \) uniformly on \( A \). Let \( \varepsilon > 0 \) and choose \( n_0 \) such that \( 1/n_0 < \varepsilon \).
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Proof (continued).

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\[
|f_k - f| < 1/n_0 \quad \text{on} \quad A_{n_0} \quad \text{for} \quad k \geq N(n_0).
\]
Since \( A \subset A_{n_0} \) and \( 1/n_0 < \varepsilon \) then the previous observation implies \( |f_k - f| < \varepsilon \) on \( A \) for \( k \geq N(n_0) \). So \( \{f_n\} \) converges to \( f \) uniformly on \( A \) and \( m(E \setminus A) < \varepsilon/2 \).
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Proof (continued).

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Now we need to find the desired closed set. By Theorem 2.11 there is a closed set \( F \) contained in \( A \) for which \( m(A \setminus F) < \varepsilon/2 \).
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Proof (continued).

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\[ m(E \setminus F) = m(E \setminus A) + m(A \setminus F) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \]

Since \( F \subset A \), then \( \{f_n\} \) converges uniformly on \( F \). \( \square \)
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Proof (continued).

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Proof. Let $a_1, a_2, \ldots, a_n$ be the finite number of distinct values taken by $f$ and let the values be taken on the sets $E_1, E_2, \ldots, E_n$ respectively. Since the $a_k$’s are distinct then the $E_k$’s are disjoint.
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Proposition 3.11. Let $f$ be a simple function defined on $E$. Then for each $\varepsilon > 0$, there is a continuous function $g$ on $\mathbb{R}$ and a closed set $F$ contained in $E$ for which

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Proof (continued). Define $g$ on $F$ as $g(x) = a_k$ for $x \in F_k$. (The $F_k$'s are disjoint, so $g$ is well-defined.) Since the $F_k$'s are closed, $g$ is continuous on $F$ (for $x \in F_k$, there is an open interval containing $x$ which is disjoint from the other $F_k$'s, so $g$ is constant on this open interval intersecting $F$).
Proposition 3.11. Let \( f \) be a simple function defined on \( E \). Then for each \( \varepsilon > 0 \), there is a continuous function \( g \) on \( \mathbb{R} \) and a closed set \( F \) contained in \( E \) for which

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Proposition 3.11. Let \( f \) be a simple function defined on \( E \). Then for each \( \varepsilon > 0 \), there is a continuous function \( g \) on \( \mathbb{R} \) and a closed set \( F \) contained in \( E \) for which

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Lusin’s Theorem

Lusin’s Theorem. Let $f$ be a real-valued measurable function on $E$. Then for each $\varepsilon > 0$, there is a continuous function $g$ on $\mathbb{R}$ and a closed set $F$ contained in $E$ for which

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Proof. The case $m(E) = \infty$ is Problem 3.29, so we consider $m(E) < \infty$. 

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Proof. The case $m(E) = \infty$ is Problem 3.29, so we consider $m(E) < \infty$. By the Simple Approximation Theorem, there is a sequence $\{f_n\}$ of simple functions defined on $E$ that converges to $f$ pointwise on $E$. Let $n \in \mathbb{N}$. 
Lusin’s Theorem. Let \( f \) be a real-valued measurable function on \( E \). Then for each \( \varepsilon > 0 \), there is a continuous function \( g \) on \( \mathbb{R} \) and a closed set \( F \) contained in \( E \) for which

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f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.
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Proof. The case \( m(E) = \infty \) is Problem 3.29, so we consider \( m(E) < \infty \). By the Simple Approximation Theorem, there is a sequence \( \{f_n\} \) of simple functions defined on \( E \) that converges to \( f \) pointwise on \( E \). Let \( n \in \mathbb{N} \). By Proposition 3.11, with \( f \) replaced by \( f_n \) and \( \varepsilon \) replaced by \( \varepsilon/2^{n+1} \), there is a continuous \( g_n \) defined on \( \mathbb{R} \) and a closed set \( F_n \) contained in \( E \) for which \( f_n = g_n \) on \( F_n \) and \( m(E \setminus F_n) < \varepsilon/2^{n+1} \).
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**Proof.** The case $m(E) = \infty$ is Problem 3.29, so we consider $m(E) < \infty$. By the Simple Approximation Theorem, there is a sequence $\{f_n\}$ of simple functions defined on $E$ that converges to $f$ pointwise on $E$. Let $n \in \mathbb{N}$. By Proposition 3.11, with $f$ replaced by $f_n$ and $\varepsilon$ replaced by $\varepsilon/2^{n+1}$, there is a continuous $g_n$ defined on $\mathbb{R}$ and a closed set $F_n$ contained in $E$ for which $f_n = g_n$ on $F_n$ and $m(E \setminus F_n) < \varepsilon/2^{n+1}$. By Egoroff’s Theorem (this is where finite measure is used), there is a closed set $F_0$ contained in $E$ such that $\{f_n\}$ converges to $f$ uniformly on $F_0$ and $m(E \setminus F_0) < \varepsilon/2$. Define $F = \bigcap_{n=0}^{\infty} F_n$. 


Lusin’s Theorem. Let \( f \) be a real-valued measurable function on \( E \). Then for each \( \varepsilon > 0 \), there is a continuous function \( g \) on \( \mathbb{R} \) and a closed set \( F \) contained in \( E \) for which

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f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.
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Proof. The case \( m(E) = \infty \) is Problem 3.29, so we consider \( m(E) < \infty \). By the Simple Approximation Theorem, there is a sequence \( \{f_n\} \) of simple functions defined on \( E \) that converges to \( f \) pointwise on \( E \). Let \( n \in \mathbb{N} \). By Proposition 3.11, with \( f \) replaced by \( f_n \) and \( \varepsilon \) replaced by \( \varepsilon/2^{n+1} \), there is a continuous \( g_n \) defined on \( \mathbb{R} \) and a closed set \( F_n \) contained in \( E \) for which \( f_n = g_n \) on \( F_n \) and \( m(E \setminus F_n) < \varepsilon/2^{n+1} \). By Egoroff’s Theorem (this is where finite measure is used), there is a closed set \( F_0 \) contained in \( E \) such that \( \{f_n\} \) converges to \( f \) uniformly on \( F_0 \) and \( m(E \setminus F_0) < \varepsilon/2 \). Define \( F = \cap_{n=0}^\infty F_n \).
Proof (continued). Then

\[
m(E \setminus F) = m(E \setminus \bigcap_{n=0}^{\infty} F_n) = m(\bigcup_{n=0}^{\infty} (E \setminus F_n)) = m((E \setminus F_0) \cup (\bigcup_{n=1}^{\infty} (E \setminus F_n))) < \varepsilon + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

The set \(F\) is closed (since it’s the intersection of closed sets \(F_n\)). Each \(f_n\) is continuous on \(F\) since \(F \subset F_n\) and \(f_n = g_n\) on \(F_n\) and \(g_n\) is continuous on \(\mathbb{R}\). Finally, \(\{f_n\}\) converges to \(f\) uniformly on \(F\) since \(F \subset F_0\) and \(\{f_n\}\) converges uniformly to \(f\) on \(F_0\) (that’s how \(F_0\) was chosen).
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Proof (continued). Then

\[ m(E \setminus F) = m(E \setminus \bigcap_{n=0}^{\infty} F_n) = m\left(\bigcup_{n=0}^{\infty} (E \setminus F_n)\right) = m((E \setminus F_0) \cup \bigcup_{n=1}^{\infty} (E \setminus F_n)) \]

\[ < \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

The set \( F \) is closed (since it’s the intersection of closed sets \( F_n \)). Each \( f_n \) is continuous on \( F \) since \( F \subseteq F_n \) and \( f_n = g_n \) on \( F_n \) and \( g_n \) is continuous on \( \mathbb{R} \). Finally, \( \{f_n\} \) converges to \( f \) uniformly on \( F \) since \( F \subseteq F_0 \) and \( \{f_n\} \) converges uniformly to \( f \) on \( F_0 \) (that’s how \( F_0 \) was chosen). However, the uniform limit of continuous functions is continuous, so the restriction of \( f \) to set \( F \) is continuous. By Problem 3.25, there is a continuous function \( g \) defined on all of \( \mathbb{R} \) such that \( g = f \) on \( F \). Function \( g \) is the desired function.
Proof (continued). Then

\[ m(E \setminus F) = m(E \setminus \bigcap_{n=0}^{\infty} F_n) = m(\bigcup_{n=0}^{\infty} (E \setminus F_n)) = m((E \setminus F_0) \cup (\bigcup_{n=1}^{\infty} (E \setminus F_n))) \]

\[ < \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

The set \( F \) is closed (since it’s the intersection of closed sets \( F_n \)). Each \( f_n \) is continuous on \( F \) since \( F \subseteq F_n \) and \( f_n = g_n \) on \( F_n \) and \( g_n \) is continuous on \( \mathbb{R} \). Finally, \( \{f_n\} \) converges to \( f \) uniformly on \( F \) since \( F \subseteq F_0 \) and \( \{f_n\} \) converges uniformly to \( f \) on \( F_0 \) (that’s how \( F_0 \) was chosen). However, the uniform limit of continuous functions is continuous, so the restriction of \( f \) to set \( F \) is continuous. By Problem 3.25, there is a continuous function \( g \) defined on all of \( \mathbb{R} \) such that \( g = f \) on \( F \). Function \( g \) is the desired function.