Theorem 4.9. Let $f$ be a nonnegative measurable function on set $E$. Then $\int_E f = 0$ if and only if $f = 0$ a.e. on $E$.

Proof. (1) Suppose $\int_E f = 0$. Then by Chebychev’s Inequality, for each $n \in \mathbb{N}$, $m(\{x \in E \mid f(x) > 1/n\}) = 0$. By Continuity of Measure (Theorem 2.15)

$$m(\{x \in E \mid f(x) > 0\}) = m(\bigcup_{n=1}^{\infty} \{x \in E \mid f(x) > 1/n\}) = \lim_{n \to \infty} m(\{x \in E \mid f(x) > 1/n\}) = 0.$$

(2) Suppose $f = 0$ a.e. on $E$. Let $\varphi$ be a simple function and $h$ a bounded measurable function of finite support for which $0 \leq \varphi \leq h \leq f$ on $E$. Then $\varphi = 0$ a.e. on $E$ and so $\int_E \varphi = 0$. Since this holds for all such $\varphi$, we have that $\int_E h = 0$. Since this holds for all such $h$, we have that $\int_E f = 0$. □
**Theorem 4.10**

**Theorem 4.10. Linearity and Monotonicity of Integration.**
Let $f$ and $g$ be nonnegative measurable functions on $E$. Then for any $\alpha > 0$ and $\beta > 0$,

$$
\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.
$$

Moreover, if $f \leq g$ on $E$ then $\int_E f \leq \int_E g$.

**Proof.** For $\alpha > 0$, $0 \leq h \leq f$ on $E$ if and only if $0 \leq \alpha h \leq \alpha f$ on $E$. Therefore

$$
\int_E \alpha f = \sup \left\{ \int_E \alpha h \mid \alpha h \text{ bounded, finite support, } 0 \leq \alpha h \leq \alpha f \right\}
= \alpha \sup \left\{ \int_E h \mid h \text{ bounded, finite support, } 0 \leq h \leq f \right\}
= \alpha \int_E f.
$$

**Theorem 4.10 (continued 2)**

**Proof (continued).** Both $h$ and $k$ are bounded measurable functions of finite support. We have $0 \leq h \leq f$, $0 \leq k \leq g$, and $\ell = h + k$ on $E$. By linearity of the integral (Theorem 4.5),

$$
\int_E \ell = \int_E h + \int_E k \leq \int_E f + \int_E g.
$$

Taking a suprema over all such $\ell$ gives $\int_E (f + g) \leq \int_E f + \int_E g$ and linearity follows.

For monotonicity, let $h$ be an arbitrary bounded measurable function of finite support for which $0 \leq h \leq f$ on $E$. Since $f \leq g$ on $E$ and so

$$
\int_E h \leq \sup \left\{ \int_E h \mid h \leq g \right\} = \int_E g.
$$

Taking a supremum over all such $h \leq f$ gives $\int_E f \leq \int_E g$.

**Theorem 4.11. Additivity Over Domain of Integration.**
Let $f$ be a nonnegative measurable function on $E$. If $A$ and $B$ are disjoint measurable subsets of $E$, then

$$
\int_{A \cup B} f = \int_A f + \int_B f.
$$

In particular, if $E_0$ is a subset of $E$ of measure zero, then $\int_E f = \int_{E \setminus E_0} f$.

**Proof.** First, for $E_1$ a measurable subset of $E$ we have

$$
\int_{E_1} f = \sup \left\{ \int_{E_1} h \mid h \text{ is bounded, measurable, } 0 \leq h \leq f \text{ on } E_1 \right\}
= \sup \left\{ \int_{E_1} h \cdot \chi_{E_1} \mid h \text{ is bounded, measurable, } 0 \leq h \cdot \chi_{E_1} \leq f \text{ on } E \right\}
$$

by Problem 4.10.
**Theorem 4.11 (continued 1)**

**Proof (continued).**

\[
\int_{E_1} f = \sup \left\{ \int_{E} h \cdot \chi_{E_1} \mid h \text{ is bounded, measurable, of finite support, and } 0 \leq h \cdot \chi_{E_1} \leq f \text{ on } E \right\}
\]

\[
= \sup \left\{ \int_{E} h \cdot \chi_{E_1} \mid h \text{ is bounded, measurable, of finite support, and } 0 \leq h \cdot \chi_{E_1} \leq f \cdot \chi_{E_1} \text{ on } E \right\}
\]

\[
= \int_{E} f \cdot \chi_{E_1}.
\]

Since \(A\) and \(B\) are disjoint then \(f \cdot \chi_{A \cup B} = f \cdot \chi_A + f \cdot \chi_B\). So by linearity (Theorem 4.10) we have

\[
\int_{A \cup B} f = \int_{E} f \cdot \chi_{A \cup B} = \int_{E} (f \cdot \chi_A + f \cdot \chi_B) = \int_{E} f \cdot \chi_A + \int_{E} f \cdot \chi_B = \int_{A} f + \int_{B} f,
\]

as claimed.

---

**Fatou's Lemma**

**Fatou's Lemma.** Let \(\{f_n\}\) be a sequence of nonnegative measurable functions on \(E\). If \(\{f_n\} \to f\) pointwise a.e. on \(E\), then

\[
\int_{E} f \leq \liminf \left( \int_{E} f_n \right).
\]

**Proof.** It follows from Theorem 4.11, that the convergence is everywhere WLOG. So \(f\) is nonnegative and measurable (by Proposition 3.9). Let \(h\) be a bounded measurable function of finite support for \(0 \leq h \leq f\) on \(E\). Choose \(M \geq 0\) for which \(|h| \leq M\) on \(E\). Define \(E_0 = \{x \in E \mid h(x) \neq 0\}\). Then \(m(E_0) < \infty\) since \(h\) is of finite support. Let \(n \in \mathbb{N}\). Define \(h_n\) on \(E\) as \(h_n = \min\{h, f_n\}\). Then \(h_n\) is measurable (by Proposition 3.8), \(0 \leq h_n \leq M\) on \(E_0\) and \(h_n = 0\) on \(E \setminus E_0\) (since \(h = 0\) there). Also, for each \(x \in E\), since \(h(x) \leq f(x)\) and \(\{f_n(x)\} \to f(x)\), then \(h_n(x) \to h(x)\).

---

**Theorem 4.11 (continued 2)**

**Theorem 4.11. Additivity Over Domain of Integration.**

Let \(f\) be a nonnegative measurable function on \(E\). If \(A\) and \(B\) are disjoint measurable subsets of \(E\), then

\[
\int_{A \cup B} f = \int_{A} f + \int_{B} f.
\]

In particular, if \(E_0\) is a subset of \(E\) of measure zero, then \(\int_{E} f = \int_{E \setminus E_0} f\).

**Proof (continued).** By Proposition 4.9, \(\int_{E_0} f = 0\) since \(m(E_0) = 0\). By additivity from above,

\[
\int_{E} f = \int_{E \setminus E_0} f + \int_{E_0} f = \int_{E \setminus E_0} f,
\]

as claimed.

---

**Fatou's Lemma (continued)**

**Proof (continued).** Applying the Bounded Convergence Theorem to \(\{h_n\}\),

\[
\lim_{n \to \infty} \left( \int_{E} h_n \right) = \lim_{n \to \infty} \left( \int_{E_0} h_n \right) = \int_{E_0} \left( \lim_{n \to \infty} h_n \right) = \int_{E_0} h = \int_{E} h.
\]

Since \(h_n \leq f_n\) on \(E\) and \(h_n\) is bounded and of finite support, by the definition of \(\int_{E} f_n, \int_{E} h_n \leq \int_{E} f_n\). Therefore

\[
\int_{E} h = \lim_{n \to \infty} \left( \int_{E} h_n \right) \leq \liminf \left( \int_{E} f_n \right).
\]

Since \(h\) is an arbitrary bounded function of finite support and \(h \leq f\), then

\[
\int_{E} f = \sup \left\{ \int_{E} h \mid h \text{ bounded, finite support, } 0 \leq h \leq f \right\} \leq \liminf \left( \int_{E} f_n \right).
\]
Monotone Convergence Theorem

**Monotone Convergence Theorem.** Let \( \{f_n\} \) be an increasing sequence of nonnegative measurable functions on \( E \). If \( \{f_n\} \to f \) pointwise a.e. on \( E \), then

\[
\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f.
\]

**Proof.** Since the sequence \( \{f_n\} \) is increasing, then \( f_n \leq f \) almost everywhere on \( E \). So by the monotonicity of integration (Theorem 4.10), \( \int_E f_n \leq \int_E f \). Therefore \( \limsup (\int_E f_n) \leq \int_E f \). By Fatou’s Lemma, \( \int_E f \leq \liminf (\int_E f_n) \). Since \( \limsup \geq \liminf \), it follows that

\[
\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E f.
\]

Corollary 4.12

**Corollary 4.12.** Let \( \{u_n\} \) be a sequence of nonnegative measurable functions on \( E \). If \( f = \sum_{n=1}^{\infty} u_n \) pointwise a.e. on \( E \), then

\[
\int_E f = \sum_{n=1}^{\infty} \left( \int_E u_n \right).
\]

**Proof.** Since each \( u_n \) is nonnegative, then \( \sum_{n=1}^{k} u_n \) is an increasing sequence of nonnegative measurable functions. So

\[
\int_E f = \lim_{k \to \infty} \int_E \sum_{n=1}^{k} u_n = \lim_{k \to \infty} \sum_{n=1}^{k} \int_E u_n \text{ by linearity (Theorem 4.10)}
\]

\[
= \sum_{n=1}^{\infty} \int_E u_n,
\]

as claimed.

Proposition 4.13

**Proposition 4.13.** Let nonnegative \( f \) be integrable over \( E \). Then \( f \) is finite a.e. on \( E \).

**Proof.** Let \( n \in \mathbb{N} \). By monotonicity of measure and Chebychev’s Inequality,

\[
m(\{x \in E | f(x) = \infty\}) \leq m(\{x \in E | f(x) \geq n\}) \leq \frac{1}{n} \int_E f.
\]

Since \( \int_E f < \infty \) and this holds for all \( n \in \mathbb{N} \), it must be that

\[
m(\{x \in E | f(x) = \infty\}) = 0.
\]

\[\square\]
Beppo Levi’s Lemma. Let \( \{f_n\} \) be an increasing sequence of nonnegative measurable functions on \( E \). If the sequence \( \{\int_E f_n\} \) is bounded, then \( \{f_n\} \) converges pointwise on \( E \) to a measurable function \( f \) that is finite a.e. on \( E \) and
\[
\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E f < \infty.
\]

Proof. Every monotone sequence of extended real numbers converges to an extended real number. So \( \{f_n\} \) converges pointwise on \( E \) and is measurable (by Proposition 3.8). By the Monotone Convergence Theorem, \( \{\int_E f_n\} \to \int_E f \). Since \( \{\int_E f_n\} \) is bounded, its limit is finite and so \( \int_E f < \infty \). By Proposition 4.13, \( f \) is finite a.e. on \( E \). ∎