Chapter 4. Lebesgue Integration
4.5. Countable Additivity and Continuity of Integration—Proofs of Theorems
Theorem 4.20. The Countable Additivity of Integration
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Let $f$ be integrable over $E$ and $\{E_n\}_{n=1}^{\infty}$ a disjoint collection of measurable subsets of $E$ whose union is $E$. Then

$$\int_E f = \sum_{n=1}^{\infty} \left( \int_{E_n} f \right).$$

Proof. Let $n \in \mathbb{N}$ and define $f_n = f \cdot \chi_n$ where $\chi_n$ is the characteristic function on $\bigcup_{k=1}^{n} E_k$. The $f_n$ are measurable and integrable by Problem 4.28.
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**Proof.** Let $n \in \mathbb{N}$ and define $f_n = f \cdot \chi_n$ where $\chi_n$ is the characteristic function on $\bigcup_{k=1}^{n} E_k$. The $f_n$ are measurable and integrable by Problem 4.28. Also $|f_n| \leq |f|$ on $E$ (in fact, $f_n(x) = f(x)$ for $x \in \bigcup_{k=1}^{n} E_k$ and $f_n(x) = 0$ for $x \notin \bigcup_{k=1}^{n} E_k$). So by the Lebesgue Dominated Convergence Theorem, $\int_E f = \int_E (\lim_{n \to \infty} f_n) = \lim_{n \to \infty} (\int_E f_n)$. 
Theorem 4.20. The Countable Additivity of Integration.

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Proof. Let $n \in \mathbb{N}$ and define $f_n = f \cdot \chi_n$ where $\chi_n$ is the characteristic function on $\bigcup_{k=1}^{n} E_k$. The $f_n$ are measurable and integrable by Problem 4.28. Also $|f_n| \leq |f|$ on $E$ (in fact, $f_n(x) = f(x)$ for $x \in \bigcup_{k=1}^{n} E_k$ and $f_n(x) = 0$ for $x \not\in \bigcup_{k=1}^{n} E_k$). So by the Lebesgue Dominated Convergence Theorem, $\int_E f = \int_E (\lim_{n \to \infty} f_n) = \lim_{n \to \infty} (\int_E f_n)$. Since the $E_n$'s are disjoint, by additivity (Corollary 4.18) we have for each $n$ that

$$\int_E f_n = \sum_{k=1}^{n} (\int_{E_k} f_n) = \sum_{k=1}^{n} (\int_{E_k} f)$$ since $f = f_n$ on $E_k$. Therefore

$$\int_E f = \lim_{n \to \infty} \left( \int_E f_n \right) = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \int_{E_k} f \right) = \sum_{k=1}^{\infty} \left( \int_{E_k} f \right).$$
Theorem 4.20. The Countable Additivity of Integration.

Let $f$ be integrable over $E$ and $\{E_n\}_{n=1}^\infty$ a disjoint collection of measurable subsets of $E$ whose union is $E$. Then

$$\int_E f = \sum_{n=1}^\infty \left( \int_{E_n} f \right).$$

Proof. Let $n \in \mathbb{N}$ and define $f_n = f \cdot \chi_n$ where $\chi_n$ is the characteristic function on $\bigcup_{k=1}^n E_k$. The $f_n$ are measurable and integrable by Problem 4.28. Also $|f_n| \leq |f|$ on $E$ (in fact, $f_n(x) = f(x)$ for $x \in \bigcup_{k=1}^n E_k$ and $f_n(x) = 0$ for $x \not\in \bigcup_{k=1}^n E_k$). So by the Lebesgue Dominated Convergence Theorem, $\int_E f = \int_E (\lim_{n \to \infty} f_n) = \lim_{n \to \infty} (\int_E f_n)$. Since the $E_n$'s are disjoint, by additivity (Corollary 4.18) we have for each $n$ that $\int_E f_n = \sum_{k=1}^n (\int_{E_k} f_n) = \sum_{k=1}^n (\int_{E_k} f)$ since $f = f_n$ on $E_k$. Therefore

$$\int_E f = \lim_{n \to \infty} \left( \int_E f_n \right) = \lim_{n \to \infty} \left( \sum_{k=1}^n \int_{E_k} f \right) = \sum_{k=1}^\infty \left( \int_{E_k} f \right).$$