Chapter 5. Lebesgue Integration: Further Topics
5.1. Uniform Integrability and Tightness: A General Vitali Convergence Theorem—Proofs of Theorems
1 Proposition 5.1

2 The General Vitali Convergence Theorem
Proposition 5.1. Let $f$ be integrable over $E$. Then for each $\varepsilon > 0$, there is a set of finite measure $E_0$ for which $\int_{E \setminus E_0} |f| < \varepsilon$.

Proof. Let $\varepsilon > 0$. The fact that $f$ is integrable over $E$ implies (by definition) that $|f|$ is integrable over $E$. By the definition of $\int_E |f|$, there is a bounded measurable function $g$ on $E$, which vanishes outside a set $E_0 \subseteq E$ of finite measure for which $0 \leq g \leq |f|$ and $\int_E |f| - \int_E g < \varepsilon$ (since $\int_E |f|$ is defined as in terms of a supremum involving integrals of such functions).
Proposition 5.1. Let $f$ be integrable over $E$. Then for each $\varepsilon > 0$, there is a set of finite measure $E_0$ for which $\int_{E \setminus E_0} |f| < \varepsilon$.

**Proof.** Let $\varepsilon > 0$. The fact that $f$ is integrable over $E$ implies (by definition) that $|f|$ is integrable over $E$. By the definition of $\int_E |f|$, there is a bounded measurable function $g$ on $E$, which vanishes outside a set $E_0 \subseteq E$ of finite measure for which $0 \leq g \leq |f|$ and $\int_E |f| - \int_E g < \varepsilon$ (since $\int_E |f|$ is defined as in terms of a supremum involving integrals of such functions). So

$$\int_{E \setminus E_0} |f| = \int_{E \setminus E_0} (|f| - g) \text{ since } g \text{ vanishes outside } E_0$$

$$\leq \int_{E \setminus E_0} (|f| - g) + \int_{E_0} (|f| - g) \text{ since } |f| - g \text{ is nonnegative}$$

$$= \int_E (|f| - g) < \varepsilon \text{ by additivity, Theorem 4.11, as claimed.} \quad \square$$
**Proposition 5.1.** Let $f$ be integrable over $E$. Then for each $\varepsilon > 0$, there is a set of finite measure $E_0$ for which $\int_{E \setminus E_0} |f| < \varepsilon$.

**Proof.** Let $\varepsilon > 0$. The fact that $f$ is integrable over $E$ implies (by definition) that $|f|$ is integrable over $E$. By the definition of $\int_E |f|$, there is a bounded measurable function $g$ on $E$, which vanishes outside a set $E_0 \subseteq E$ of finite measure for which $0 \leq g \leq |f|$ and $\int_E |f| - \int_E g < \varepsilon$ (since $\int_E |f|$ is defined as in terms of a supremum involving integrals of such functions). So

$$\int_{E \setminus E_0} |f| = \int_{E \setminus E_0} (|f| - g) \text{ since } g \text{ vanishes outside } E_0$$

$$\leq \int_{E \setminus E_0} (|f| - g) + \int_{E_0} (|f| - g) \text{ since } |f| - g \text{ is nonnegative}$$

$$= \int_E (|f| - g) < \varepsilon \text{ by additivity, Theorem 4.11, as claimed.} \qed$$
The General Vitali Convergence Theorem.

Let \( \{f_n\} \) be a sequence of functions on \( E \) that is uniformly integrable and tight over \( E \). Suppose \( \{f_n\} \to f \) pointwise a.e. on \( E \). Then \( f \) is integrable over \( E \) and

\[
\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f.
\]

Proof. Let \( \varepsilon > 0 \). Since \( \{f_n\} \) is tight over \( E \), there is measurable \( E_0 \subseteq E \) of finite measure for which \( \int_{E \setminus E_0} |f_n| < \varepsilon/4 \) for all \( n \in \mathbb{N} \).
The General Vitali Convergence Theorem.

Let \( \{f_n\} \) be a sequence of functions on \( E \) that is uniformly integrable and tight over \( E \). Suppose \( \{f_n\} \to f \) pointwise a.e. on \( E \). Then \( f \) is integrable over \( E \) and

\[
\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f.
\]

**Proof.** Let \( \varepsilon > 0 \). Since \( \{f_n\} \) is tight over \( E \), there is measurable \( E_0 \subseteq E \) of finite measure for which \( \int_{E \setminus E_0} |f_n| < \varepsilon/4 \) for all \( n \in \mathbb{N} \). By Fatou’s Lemma, \( \int_{E \setminus E_0} |f| \leq \liminf \left( \int_{E \setminus E_0} |f_n| \right) \leq \varepsilon/4 \). So \( f \) is integrable over \( E \setminus E_0 \).
The General Vitali Convergence Theorem.

Let \( \{f_n\} \) be a sequence of functions on \( E \) that is uniformly integrable and tight over \( E \). Suppose \( \{f_n\} \rightarrow f \) pointwise a.e. on \( E \). Then \( f \) is integrable over \( E \) and

\[
\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f.
\]

Proof. Let \( \varepsilon > 0 \). Since \( \{f_n\} \) is tight over \( E \), there is measurable \( E_0 \subseteq E \) of finite measure for which \( \int_{E \setminus E_0} |f_n| < \varepsilon/4 \) for all \( n \in \mathbb{N} \). By Fatou’s Lemma, \( \int_{E \setminus E_0} |f| \leq \lim\inf \left( \int_{E \setminus E_0} |f_n| \right) \leq \varepsilon/4 \). So \( f \) is integrable over \( E \setminus E_0 \). Also

\[
\left| \int_{E \setminus E_0} (f_n - f) \right| \leq \int_{E \setminus E_0} |f_n - f| \text{ by Proposition 4.16, Integral Comparison Test.}
\]
The General Vitali Convergence Theorem

Let \( \{f_n\} \) be a sequence of functions on \( E \) that is uniformly integrable and tight over \( E \). Suppose \( \{f_n\} \rightarrow f \) pointwise a.e. on \( E \). Then \( f \) is integrable over \( E \) and

\[
\lim_{n \rightarrow \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \rightarrow \infty} f_n \right) = \int_E f.
\]

Proof. Let \( \varepsilon > 0 \). Since \( \{f_n\} \) is tight over \( E \), there is measurable \( E_0 \subseteq E \) of finite measure for which \( \int_{E \setminus E_0} |f_n| < \varepsilon / 4 \) for all \( n \in \mathbb{N} \). By Fatou’s Lemma, \( \int_{E \setminus E_0} |f| \leq \lim \inf \left( \int_{E \setminus E_0} |f_n| \right) \leq \varepsilon / 4 \). So \( f \) is integrable over \( E \setminus E_0 \). Also

\[
\left| \int_{E \setminus E_0} (f_n - f) \right| \leq \int_{E \setminus E_0} |f_n - f| \text{ by Proposition 4.16, Integral Comparison Test...}
\]
The General Vitali Convergence Theorem (continued 1)

Proof (continued). . .

\[ \left| \int_{E \setminus E_0} (f_n - f) \right| \leq \int_{E \setminus E_0} |f_n - f| \]
\[ \leq \int_{E \setminus E_0} |f_n| + \int_{E \setminus E_0} |f| \text{ by the Triangle Inequality and monotonicity (Theorem 4.10)} \]
\[ < \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \]

Since \( m(E_0) < \infty \) and \( \{f_n\} \) is uniformly integrable over \( E_0 \) then each \( f_n \) is integrable over \( E_0 \) by Proposition 4.23, and by the Vitali Convergence Theorem \( f \) is integrable over \( E_0 \) and \( \lim_{n \to \infty} \left( \int_{E_0} f_n \right) = \int_{E_0} f. \)
The General Vitali Convergence Theorem (continued 1)

Proof (continued). . .

\[ \left| \int_{E \setminus E_0} (f_n - f) \right| \leq \int_{E \setminus E_0} |f_n - f| \]
\[ \leq \int_{E \setminus E_0} |f_n| + \int_{E \setminus E_0} |f| \text{ by the Triangle Inequality} \]
\[ \text{and monotonicity (Theorem 4.10)} \]
\[ < \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \]

Since \( m(E_0) < \infty \) and \( \{f_n\} \) is uniformly integrable over \( E_0 \) then each \( f_n \) is integrable over \( E_0 \) by Proposition 4.23, and by the Vitali Convergence Theorem \( f \) is integrable over \( E_0 \) and \( \lim_{n \to \infty} \left( \int_{E_0} f_n \right) = \int_{E_0} f. \) So there is \( N \in \mathbb{N} \) such that \( \left| \int_{E_0} f_n - \int_{E_0} f \right| = \left| \int_{E_0} (f_n - f) \right| < \varepsilon/2 \) for all \( n \geq N \) (by linearity, Theorem 4.17). Since \( f \) is integrable over \( E \setminus E_0 \) and \( E_0 \), then \( f \) is integrable over \( E \). Each \( f_n \) is integrable over \( E \) by Note 5.1.A.
The General Vitali Convergence Theorem (continued 1)

Proof (continued). . . .

\[
\left| \int_{E \setminus E_0} (f_n - f) \right| \leq \int_{E \setminus E_0} |f_n - f| \\
\leq \int_{E \setminus E_0} |f_n| + \int_{E \setminus E_0} |f| \text{ by the Triangle Inequality and monotonicity (Theorem 4.10)} \\
< \varepsilon/4 + \varepsilon/4 = \varepsilon/2.
\]

Since \( m(E_0) < \infty \) and \( \{f_n\} \) is uniformly integrable over \( E_0 \) then each \( f_n \) is integrable over \( E_0 \) by Proposition 4.23, and by the Vitali Convergence Theorem \( f \) is integrable over \( E_0 \) and \( \lim_{n \to \infty} \left( \int_{E_0} f_n \right) = \int_{E_0} f \). So there is \( N \in \mathbb{N} \) such that \( \left| \int_{E_0} f_n - \int_{E_0} f \right| = \left| \int_{E_0} (f_n - f) \right| < \varepsilon/2 \) for all \( n \geq N \) (by linearity, Theorem 4.17). Since \( f \) is integrable over \( E \setminus E_0 \) and \( E_0 \), then \( f \) is integrable over \( E \). Each \( f_n \) is integrable over \( E \) by Note 5.1.A.
Proof (continued). Combining the above,

\[ \left| \int_E f_n - \int_E f \right| = \left| \int_E (f_n - f) \right| \quad \text{by linearity, Theorem 4.17} \]

\[ = \left| \int_{E \setminus E_0} (f_n - f) + \int_{E_0} (f_n - f) \right| \quad \text{by additivity, Cor. 4.18} \]

\[ \leq \left| \int_{E \setminus E_0} (f_n - f) \right| + \left| \int_{E_0} (f_n - f) \right| \quad \text{by Triangle Inequality} \]

\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{for } n \geq N \]

\[ = \varepsilon. \]

So \( \lim_{n \to \infty} (\int_E f_n) = \int_E f. \)