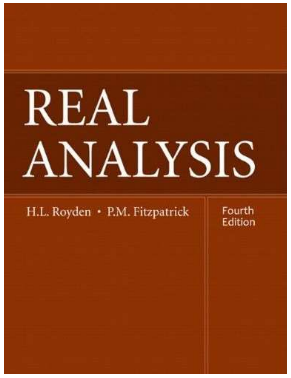
Real Analysis

Chapter 9. Metric Spaces: General Properties

9.2. Open Sets, Closed Sets, and Convergent Sequences—Proofs of Theorems



Proposition 9.1

Proposition 9.1. Let X be a metric space. Then sets X and $\varnothing$ are open. The intersection of any two open subsets of X is open. The union of any collection of open subsets is open.

Proof. Set X is open trivially. Set $\varnothing$ is open vacuously. If $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ is a collection of open sets, then for $x \in \cup_{\alpha \in A} \mathcal{O}_\alpha$ we have $x \in \mathcal{O}_{\alpha'}$ for any given $\alpha' \in A$, so there is $r > 0$ such that $B(x, r) \subset \mathcal{O}_{\alpha'}$ since $\mathcal{O}_{\alpha'}$ is open. Also, $B(x, r) \subset \cup_{\alpha \in A} \mathcal{O}_\alpha$ so $\cup_{\alpha \in A} \mathcal{O}_\alpha$ is open.

Given $\mathcal{O}_1, \mathcal{O}_2$ open, if $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$ then $\mathcal{O}_1 \cap \mathcal{O}_2$ is open. If $\mathcal{O}_1 \cap \mathcal{O}_2 \neq \varnothing$ then for any $x \in \mathcal{O}_1 \cap \mathcal{O}_2$ we have $x \in \mathcal{O}_1$ and so there is $r_1 > 0$ such that $B(x, r_1) \subset \mathcal{O}_1$ and there is $r_2 > 0$ such that $B(x, r_2) \subset \mathcal{O}_2$. With $r = \min\{r_1, r_2\}$ we have $B(x, r) \subset \mathcal{O}_1 \cap \mathcal{O}_2$ and so $\mathcal{O}_1 \cap \mathcal{O}_2$ is open. □

Proposition 9.3

Proposition 9.3. For E a subset of a metric space X, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and $E \subset F$ then $\overline{E} \subset F$.

Proof. Let x be a point of closure of $\overline{E}$. Consider a neighborhood U_x of x. Then (by the definition of "x is a point of closure of $\overline{E}$") there is $x' \in \overline{E} \cap U_x$. Since x' is a point of closure of E and U_x is a neighborhood of x' then (by the definition of "x' is a point of closure of E") there is a point $x'' \in E \cap U_x$. Therefore arbitrary neighborhood U_x of x contains a point of E and so $x \in \overline{E}$. So $\overline{E}$ contains all its points of closure and hence $\overline{E}$ is closed.

Now $A \subset B$ implies $\overline{A} \subset \overline{B}$ (every point of closure of A is a limit point of B by definition), so if F is closed and $E \subset F$ then $\overline{E} \subset \overline{F} = F$. □

Proposition 9.4

Proposition 9.4. A subset of a metric space X is open if and only if its complement in X is closed.

Proof. Suppose E is open in X. Let x be a point of closure of $X \setminus E$. Then x cannot belong to E since this would imply that there is a neighborhood of x that is contained in E and thus is disjoint from $X \setminus E$ (implying that x is *not* a point of closure of $X \setminus E$). So $x \in X \setminus E$ and $X \setminus E$ is closed.

Suppose $X \setminus E$ is closed. Let $x \in E$. Then x is not a point of closure of $X \setminus E$ (since $X \setminus E$ contains all of its points of closure) so there is a neighborhood U_x of x that does not intersect $X \setminus E$; that is, $U_x \subset E$. Then E is open (U_x contains a ball centered at x). □

Proposition 9.6

Proposition 9.6. For a subset E of a metric space X, a point $x \in X$ is a point of closure of E if and only if x is the limit of a subsequence in E. Therefore, E is closed if and only if whenever a sequence in E converges to a limit $x \in X$, the limit of x belongs to E.

Proof. First, suppose $x \in \overline{E}$. For each $n \in \mathbb{N}$ since $B(x, 1/n) \cap E \neq \varnothing$ there is $x_n \in B(x, 1/n) \cap E$. The resulting sequence satisfies $\{x_n\} \subset E$. For any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then $\rho(x_n, x) < 1/n \leq 1/N < \varepsilon$ for all $n \geq N$. So $\{x_n\} \to x$.

Conversely, if a sequence in E converges to x, then every ball centered at x contains infinitely many terms of the sequence and therefore contains points in E. So $x \in \overline{E}$. □