Real Analysis

Chapter 9. Metric Spaces: General Properties

9.2. Open Sets, Closed Sets, and Convergent Sequences—Proofs of Theorems

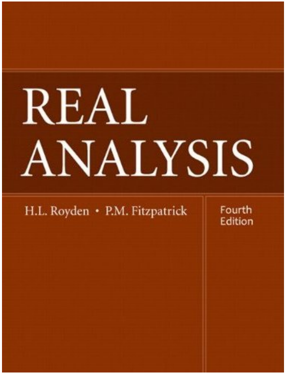

Table of contents

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Given $\mathcal{O}_1, \mathcal{O}_2$ open, if $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$ then $\mathcal{O}_1 \cap \mathcal{O}_2$ is open. If $\mathcal{O}_1 \cap \mathcal{O}_2 \neq \varnothing$ then for any $x \in \mathcal{O}_1 \cap \mathcal{O}_2$ we have $x \in \mathcal{O}_1$ and so there is $r_1 > 0$ such that $B(x, r_1) \subset \mathcal{O}_1$ and there is $r_2 > 0$ such that $B(x, r_2) \subset \mathcal{O}_2$.

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Proposition 9.3

Proposition 9.3. For E a subset of a metric space X, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and $E \subset F$ then $\overline{E} \subset F$.

Proof. Let x be a point of closure of $\overline{E}$. Consider a neighborhood U_x of x. Then (by the definition of "x is a point of closure of $\overline{E}$") there is $x' \in \overline{E} \cap U_x$.

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Proposition 9.4

Proposition 9.4. A subset of a metric space X is open if and only if its complement in X is closed.

Proof. Suppose E is open in X. Let x be a point of closure of $X \setminus E$. Then x cannot belong to E since this would imply that there is a neighborhood of x that is contained in E and thus is disjoint from $X \setminus E$ (implying that x is *not* a point of closure of $X \setminus E$). So $x \in X \setminus E$ and $X \setminus E$ is closed.

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Suppose $X \setminus E$ is closed. Let $x \in E$. Then x is not a point of closure of $X \setminus E$ (since $X \setminus E$ contains all of its points of closure) so there is a neighborhood U_x of x that does not intersect $X \setminus E$; that is, $U_x \subset E$. Then E is open (U_x contains a ball centered at x). □

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Proposition 9.6. For a subset E of a metric space X, a point $x \in X$ is a point of closure of E if and only if x is the limit of a subsequence in E. Therefore, E is closed if and only if whenever a sequence in E converges to a limit $x \in X$, the limit of x belongs to E.

Proof. First, suppose $x \in \overline{E}$. For each $n \in \mathbb{N}$ since $B(x, 1/n) \cap E \neq \varnothing$ there is $x_n \in B(x, 1/n) \cap E$. The resulting sequence satisfies $\{x_n\} \subset E$.

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