Real Analysis

Chapter 9. Metric Spaces: General Properties

9.3. Continuous Mappings Between Metric Spaces—Proofs of Theorems

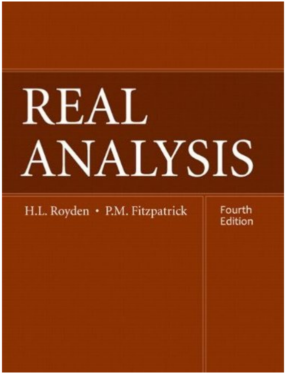

Table of contents

Theorem 9.3.A. The ε/δ Criterion for Continuity.

Theorem. The ε/δ Criterion for Continuity.
A mapping f from a metric space (X, ρ) to a metric space (Y, σ) is continuous at the point $x \in X$ if and only if for every point $\varepsilon > 0$ there is $\delta > 0$ for which if $\rho(x, x') < \delta$ then $\sigma(f(x), f(x')) < \varepsilon$; that is, $f(B(x, \delta)) \subset B(f(x), \varepsilon)$.

Proof. First, suppose $f : X \to Y$ is continuous. ASSUME there is $\varepsilon_0 > 0$ for which there is no $\delta > 0$ for which $f(B(x, \delta)) \subset B(f(x), \varepsilon_0)$. In particular if $n \in \mathbb{N}$ and $n > 1/\delta$ then it is not true that $f(B(x, 1/n)) \subset B(f(x), \varepsilon_0)$.

Theorem 9.3.A. The ε/δ Criterion for Continuity.

Theorem. The ε/δ Criterion for Continuity.
A mapping f from a metric space (X, ρ) to a metric space (Y, σ) is continuous at the point $x \in X$ if and only if for every point $\varepsilon > 0$ there is $\delta > 0$ for which if $\rho(x, x') < \delta$ then $\sigma(f(x), f(x')) < \varepsilon$; that is, $f(B(x, \delta)) \subset B(f(x), \varepsilon)$.

Proof. First, suppose $f : X \to Y$ is continuous. ASSUME there is $\varepsilon_0 > 0$ for which there is no $\delta > 0$ for which $f(B(x, \delta)) \subset B(f(x), \varepsilon_0)$. In particular if $n \in \mathbb{N}$ and $n > 1/\delta$ then it is not true that $f(B(x, 1/n)) \subset B(f(x), \varepsilon_0)$. So there is $x_n \in X$ such that $\rho(x, x_n) < 1/n$ (i.e., $x_n \in B(x, 1/n)$) while $\sigma(f(x), f(x_n)) \geq \varepsilon_0$, or $f(x_0) \notin B(f(f(x), \varepsilon_0))$. We then have sequence $\{x_n\}$ in X that converges to x, but $\{f(x_n)\}$ does not converge to $f(x)$ since $\rho(f(x), f(x_n)) \geq \varepsilon_0$ for all $n \geq 1/\delta$. But this CONTRADICTS the definition of continuity of $f : X \to Y$ at $x \in X$. So the assumption that such $\varepsilon_0 > 0$ does not exist is false and so for all $\varepsilon > 0$ there is $\delta > 0$, as claimed.

Theorem 9.3.A. The ε/δ Criterion for Continuity.

Theorem. The ε/δ Criterion for Continuity.
A mapping f from a metric space (X, ρ) to a metric space (Y, σ) is continuous at the point $x \in X$ if and only if for every point $\varepsilon > 0$ there is $\delta > 0$ for which if $\rho(x, x') < \delta$ then $\sigma(f(x), f(x')) < \varepsilon$; that is, $f(B(x, \delta)) \subset B(f(x), \varepsilon)$.

Proof. First, suppose $f : X \to Y$ is continuous. ASSUME there is $\varepsilon_0 > 0$ for which there is no $\delta > 0$ for which $f(B(x, \delta)) \subset B(f(x), \varepsilon_0)$. In particular if $n \in \mathbb{N}$ and $n > 1/\delta$ then it is not true that $f(B(x, 1/n)) \subset B(f(x), \varepsilon_0)$. So there is $x_n \in X$ such that $\rho(x, x_n) < 1/n$ (i.e., $x_n \in B(x, 1/n)$) while $\sigma(f(x), f(x_n)) \geq \varepsilon_0$, or $f(x_0) \notin B(f(f(x), \varepsilon_0))$. We then have sequence $\{x_n\}$ in X that converges to x, but $\{f(x_n)\}$ does not converge to $f(x)$ since $\rho(f(x), f(x_n)) \geq \varepsilon_0$ for all $n \geq 1/\delta$. But this CONTRADICTS the definition of continuity of $f : X \to Y$ at $x \in X$. So the assumption that such $\varepsilon_0 > 0$ does not exist is false and so for all $\varepsilon > 0$ there is $\delta > 0$, as claimed.

Theorem 9.3.A. The ε/δ Criterion for Continuity (continued).

Theorem 9.3.A. The ε/δ Criterion for Continuity.
A mapping f from a metric space (X, ρ) to a metric space (Y, σ) is continuous at the point $x \in X$ if and only if for every point $\varepsilon > 0$ there is $\delta > 0$ for which if $\rho(x, x') < \delta$ then $\sigma(f(x), f(x')) < \varepsilon$; that is, $f(B(x, \delta)) \subset B(f(x), \varepsilon)$.

Proof (continued). For the converse, suppose the ε/δ criterion holds. Let $\{x_n\}$ be a sequence in X that converges to x. Let $\varepsilon > 0$. Then there is $\delta > 0$ for which $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. Since $\{x_n\} \to x$ there is $N \in \mathbb{N}$ such that $x_n \in B(x, \delta)$ for $n \geq N$. Then $f(x_n) \in B(f(x), \varepsilon)$ for $n \geq N$; that is, $\sigma(f(x), f(x_n)) < \varepsilon$ for $n \geq N$. So $\{f(x_n)\} \to f(x)$. Since $\{x_n\}$ is an arbitrary sequence in X which converges to x, then f is continuous at x by definition. $\square$

Proposition 9.8

Proposition 9.8. A mapping f from metric space X to metric space Y is continuous if and only if for each open subset $\mathcal{O}$ of Y, the inverse image under f of $\mathcal{O}$, $f^{-1}(\mathcal{O})$, is an open subset of X.

Proof. Suppose f is continuous. Let $\mathcal{O}$ be open in Y. Let $x \in f^{-1}(\mathcal{O})$. Then $f(x) \in \mathcal{O}$ and since $\mathcal{O}$ is open there is some $r > 0$ such that $B(f(x), r) \subset \mathcal{O}$. Since f is continuous by hypothesis, then by the ε/δ criterion for continuity, there is $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), r) \subset \mathcal{O}$. Thus $B(x, \delta) \subset f^{-1}(\mathcal{O})$ and so $f^{-1}(\mathcal{O})$ is open in X.

Proposition 9.8

Proposition 9.8. A mapping f from metric space X to metric space Y is continuous if and only if for each open subset $\mathcal{O}$ of Y, the inverse image under f of $\mathcal{O}$, $f^{-1}(\mathcal{O})$, is an open subset of X.

Proof. Suppose f is continuous. Let $\mathcal{O}$ be open in Y. Let $x \in f^{-1}(\mathcal{O})$. Then $f(x) \in \mathcal{O}$ and since $\mathcal{O}$ is open there is some $r > 0$ such that $B(f(x), r) \subset \mathcal{O}$. Since f is continuous by hypothesis, then by the ε/δ criterion for continuity, there is $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), r) \subset \mathcal{O}$. Thus $B(x, \delta) \subset f^{-1}(\mathcal{O})$ and so $f^{-1}(\mathcal{O})$ is open in X.

Now suppose the inverse image under f of each open set is open. Let $x \in X$. Let $\varepsilon > 0$. The ball $B(f(x), \varepsilon)$ is open in Y. So by hypothesis $f^{-1}(B(f(x), \varepsilon))$ is open in X. So there is $\delta > 0$ with $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$. That is, $f(B(x, \delta)) \subset B(f(x), \varepsilon)$ as claimed. □

Proposition 9.8

Proposition 9.8. A mapping f from metric space X to metric space Y is continuous if and only if for each open subset $\mathcal{O}$ of Y, the inverse image under f of $\mathcal{O}$, $f^{-1}(\mathcal{O})$, is an open subset of X.

Proof. Suppose f is continuous. Let $\mathcal{O}$ be open in Y. Let $x \in f^{-1}(\mathcal{O})$. Then $f(x) \in \mathcal{O}$ and since $\mathcal{O}$ is open there is some $r > 0$ such that $B(f(x), r) \subset \mathcal{O}$. Since f is continuous by hypothesis, then by the ε/δ criterion for continuity, there is $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), r) \subset \mathcal{O}$. Thus $B(x, \delta) \subset f^{-1}(\mathcal{O})$ and so $f^{-1}(\mathcal{O})$ is open in X.

Now suppose the inverse image under f of each open set is open. Let $x \in X$. Let $\varepsilon > 0$. The ball $B(f(x), \varepsilon)$ is open in Y. So by hypothesis $f^{-1}(B(f(x), \varepsilon))$ is open in X. So there is $\delta > 0$ with $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$. That is, $f(B(x, \delta)) \subset B(f(x), \varepsilon)$ as claimed. □

Proposition 9.9

Proposition 9.9. The composition of continuous mappings between metric spaces, when defined, is continuous.

Proof. Let $f : X \to Y$ be continuous and $g : Y \to Z$ be continuous where X, Y, Z are metric spaces. Let $\mathcal{O}$ be open in Z. Since g is continuous then $g^{-1}(\mathcal{O})$ is open in Y; since f is continuous then $f^{-1}(g^{-1}(\mathcal{O})) = (g \circ f)^{-1}(\mathcal{O})$ is open in X. Therefore, by Proposition 9.8, $g \circ f$ is continuous. □

Proposition 9.9

Proposition 9.9. The composition of continuous mappings between metric spaces, when defined, is continuous.

Proof. Let $f : X \to Y$ be continuous and $g : Y \to Z$ be continuous where X, Y, Z are metric spaces. Let $\mathcal{O}$ be open in Z. Since g is continuous then $g^{-1}(\mathcal{O})$ is open in Y; since f is continuous then $f^{-1}(g^{-1}(\mathcal{O})) = (g \circ f)^{-1}(\mathcal{O})$ is open in X. Therefore, by Proposition 9.8, $g \circ f$ is continuous. □