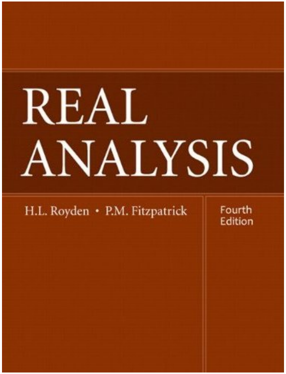
Real Analysis

Chapter 9. Metric Spaces: General Properties

9.4. Complete Metric Spaces—Proofs of Theorems



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Proposition 9.10

Proposition 9.10. Let $[a, b]$ be a closed, bounded interval of real numbers. Then $C([a, b])$, with the metric induced by the max norm, is complete.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $C[a, b]$. First, suppose there is a convergent series of real numbers $\sum_{k=1}^{\infty} a_k$ such that

$$\|f_{k+1} - f_k\|_{\max} \leq a_k \text{ for all } k. \tag{2}$$

Since $f_{n+k} - f_n = \sum_{j=n}^{n+k-1}(f_{j+1} - f_j)$ for all n, k (it behaves like a telescoping series), then by the triangle inequality in $C[a, b]$ and (2)

$$\|f_{n+k} - f_n\|_{\max} \leq \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\|_{\max} \leq \sum_{j=n}^{\infty} a_j \text{ for all } n, k.$$

Let x be an arbitrary element of $[a, b]$. Then

$$|f_{n+k}(x) - f_n(x)| \leq \|f_{k+1} - f_k\|_{\max} \leq a_k \text{ for all } k. \tag{3}$$

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$$|f_{n+k}(x) - f_n(x)| \leq \|f_{k+1} - f_k\|_{\max} \leq a_k \text{ for all } k. \qquad (3)$$

Proposition 9.10 (continued 1)

Proof (continued). Since the series $\sum_{k=1}^{\infty} a_k$ converges, then the sequence of real numbers $\{f_n(x)\}$ is a Cauchy sequence and, since the real numbers are complete, this sequence converges to some real number which we denote as $f(x)$. Since x is an arbitrary element of $[a, b]$, taking the limit as $k \to \infty$ in (3) we have

$$|f(x) - f_n(x)| \leq \sum_{j=n}^{\infty} a_j \text{ for all } n \text{ and all } x \in [a, b].$$

That is, $\{f_n\}$ converges uniformly on $[a, b]$ to f. Since each f_n is continuous, the f is also continuous and so $f \in C[a, b]$ (see Theorem 8-2 in my online notes for Analysis 1 [MATH 4217/5217] on Section 8.1. Sequences of Functions). So the result holds under the assumption of the existence of the Cauchy sequence $\sum_{k=1}^{\infty} a_k$ given above.

Proposition 9.10 (continued 2)

Proposition 9.10. Let $[a, b]$ be a closed, bounded interval of real numbers. Then $C([a, b])$, with the metric induced by the max norm, is complete.

Proof (continued). The general case follows from the fact that every Cauchy sequence $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that equation (2) holds:

$$\|f_{n_{k+1}} - f_{n_k}\|_{\max} \leq a_k \text{ for all } k$$

for some sequence of real numbers $\{a_k\}$ such that $\sum_{k=1}^{\infty} a_k$ (we can take $\varepsilon = a_k = 1/2^k$ and choose n_k based on this; then $\{a_k\}$ determines a convergent geometric series). We can then apply the above result to show that the subsequence $\{f_{n_k}\}$ converges in $C[a, b]$. A Cauchy sequence with a convergent subsequence is convergent (by Problem 9.38) so the general result holds, as claimed. □

Proposition 9.11

Proposition 9.11. If E is a subset of the complete metric space X, then the metric subspace E is complete if and only if E is a closed subset of X.

Proof. First, suppose that E is a closed subset of X. Let $\{x_n\}$ be a Cauchy sequence in E. Then $\{x_n\}$ can be considered as a Cauchy sequence in X, and X is complete by hypothesis. So $\{x_n\}$ converges to some point $x \in X$. By Proposition 9.6, since E is a closed subset of X then the limit of a convergent sequence in E belongs to E. So $x \in E$ and, since $\{x_n\}$ is an arbitrary Cauchy sequence in E then E is a complete metric space.

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Second, for the converse suppose that E is a complete metric space. By Proposition 9.6, to show that E is closed it is sufficient to show that the limit of a convergent sequence in E also belongs to E. Let $\{x_n\}$ be a sequence in E that converges to $x \in X$. But a convergent sequence is Cauchy (by the Triangle Inequality), so by the hypothesized completeness of E, $\{x_n\}$ converges to a point in E. But a convergent sequence in a metric space has only one limit, so $x \in E$. That is E is closed, as claimed. □

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Theorem 9.4.A. The Cantor Intersection Theorem

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Proof. First, assume X is a complete metric space. Let $\{F_n\}_{n=1}^{\infty}$ be a contracting sequence of nonempty closed subsets of X. For each index n, select $x_n \in F_n$ (which can be done since each F_n is nonempty). Let $\varepsilon > 0$. Since $\{F_n\}$ is a contracting sequence, then there is an index N for which $\text{diam}(F_N) < \varepsilon$. Since $\{F_n\}$ is descending, if $n, m \geq N$ then x_n and x_m belong to F_N so that $\rho(x_n, x_m) \leq \text{diam}(F_N) < \varepsilon$. Therefore $\{x_n\}$ is a Cauchy sequence.

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Theorem 9.4.A. The Cantor Intersection Theorem (cont.)

Proof (continued). Second, assume that for any contracting sequence $\{F_n\}_{n=1}^{\infty}$ of nonempty closed subsets of X, there is a point $x \in X$ for which $\cap_{n=1}^{\infty} F_n = \{x\}$. Let $\{x_n\}$ be a Cauchy sequence in X. For each index n, define F_n to be the closure of the nonempty set $\{x_k \mid k \geq n\}$. Then $\{F_n\}$ is a descending sequence of nonempty closed sets. Since $\{x_n\}$ is Cauchy (by choice), then $\text{diam}(F_n) \to 0$ as $n \to \infty$; that is, $\{F_n\}$ is contracting. So by our hypotheses in this case, there is a point $x \in X$ for which $\{x\} = \cap_{n=1}^{\infty} F_n$. Since $x \in F_n$ for each index n then x is a point of closure of $\{x_k \mid k \geq n\}$ and therefore any ball centered at x has nonempty intersection with $\{x_k \mid k \geq n\}$ (by the definition of point of closure). Hence we may inductively select a strictly increasing sequence of natural numbers $\{n_k\}$ such that for each index k, we have $\rho(x, x_{n_k}) < 1/k$ (i.e., we take $\varepsilon = 1/k$ and choose x_{n_k} accordingly). Then subsequence $\{x_{n_k}\}$ converges to x. Since $\{x_n\}$ is Cauchy and has a subsequence that converges to x, then the sequence $\{x_n\}$ converges to x (by Problem 38). Since $\{x_n\}$ is an arbitrary Cauchy sequence in X, then X is complete. □

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