Real Analysis

Chapter 9. Metric Spaces: General Properties

9.5. Compact Metric Spaces—Proofs of Theorems

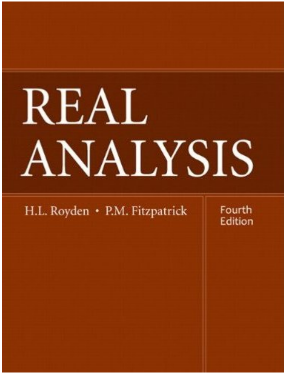

Table of contents

Lemma 9.5.A.

Lemma 9.5.A. If a metric space X is totally bounded then it is bounded in the sense that its diameter is finite.

Proof. Let $\varepsilon = 1$. Since X is totally bounded, then there are a finite number of open balls $\{B(x_k, 1)\}_{k=1}^{n}$ such that $X \subseteq \cup_{k=1}^{n} B(x_k, 1)$. Let d be the maximum distance between the centers of the open balls, $d = \max\{\rho(x_i, x_j) \mid 1 \leq i < j \leq n\}$. Then by the Triangle Inequality, $\text{diam}(X) \leq c$ where $c = 2 + d$. That is, X is bounded, as claimed. □

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Proposition 9.15

Proposition 9.15. A subset of Euclidean space $\mathbb{R}^n$ is bounded if and only if it is totally bounded.

Proof. By Lemma 9.5.A a totally bounded metric space is bounded, so if a subset of $\mathbb{R}^n$ is totally bounded then it is bounded.

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Proof. By Lemma 9.5.A a totally bounded metric space is bounded, so if a subset of $\mathbb{R}^n$ is totally bounded then it is bounded.

Now let E be a bounded subset of $\mathbb{R}^n$. Let $\varepsilon > 0$. Since E is bounded, we may take $a > 0$ large enough so that E is contained in the hypercube $[-a, a] \times [-a, a] \times \cdots \times [-a, a]$. Let P_k be a partition of $[-a, a]$ into closed intervals where each interval has length less than $1/k$ (this is possible since $[-a, a]$ is bounded). Then $P_k \times P_k \times \cdots \times P_k$ induces a partition of $[-a, a] \times [-a, a] \times \cdots \times [-a, a]$ into closed rectangles of diameter at most $\sqrt{n}/k$. Choose k such that $\sqrt{n}/k < \varepsilon$.

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Proposition 9.17

Proposition 9.17. If a metric space X is complete and totally bounded, then it is compact.

Proof. ASSUME $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is an open cover of X for which there is no finite subcover. Since X is totally bounded, we may chose a finite collection of open balls of radius less than $1/2$ that cover X. There must be one of these balls that cannot be covered by a finite subcollection of $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ (or else $\{\mathcal{O}_\lambda\}$ *does have* a finite subcover of X). Select such a ball and label its closure F_1. Then F_1 is closed and $\text{diam}(F_1) \leq 1$. Using the total boundedness of X again, there is a finite collection of open balls of radius less than $1/4$ that cover X, so so also covers F_1.

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Proposition 9.17 (continued)

Proposition 9.17. If a metric space X is complete and totally bounded, then it is compact.

Proof (continued). Continuing in this way iteratively we obtain a contracting sequence of nonempty, closed sets $\{F_n\}$ with the property that each F_n cannot be covered by a finite subcollection of $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$. But X is complete, so by the Cantor Intersection Theorem (of Section 9.4) there is a single point $x_0 \in X$ that belongs to the intersection $\cap_{n=1}^\infty F_n$. Since $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is a covering of X, there is some index λ_0 such that$\mathcal{O}_{\lambda_0}$ contains x_0 and since $\mathcal{O}_{\lambda_0}$ is open, there is a ball centered at x_0, $B(x_0, r)$, such that $B(x_0, r) \subseteq \mathcal{O}_{\lambda_0}$. Since $\lim_{n\to\infty} \text{diam}(F_n) = 0$ and $x_0 \in \cap_{n=1}^\infty F_n$, there is an index n such that $F_n \subseteq \mathcal{O}_{\lambda_0}$. This is a CONTRADICTION to the fact that each F_n was chosen as being a set that cannot be covered by a finite subcollection of $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$. So the assumption that there is an open cover of X for which there is no finite subcover is false. That is, every open cover of X has a finite subcover so that X is compact, as claimed. □

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Proposition 9.18

Proposition 9.18. If a metric space X is compact, then it is sequentially compact.

Proof. Let X be compact and let $\{x_n\}$ be a sequence in X. For each $n \in \mathbb{N}$, let F_n be the closure of the nonempty set $\{x_k \mid k \geq n\}$. Then $\{F_n\}$ is a descending sequence of nonempty closed sets which satisfy the finite intersection property. Therefore, by Proposition 9.14 there is a point $x_0 \in X$ such that $x_0 \in \cap_{n=1}^{\infty} F_n$. Since for each $n \in \mathbb{N}$, x_0 is in the closure of $\{x_k \mid k \geq n\}$, the ball $B(x_0, 1/k)$ has nonempty intersection with $\{x_k \mid k \geq n\}$.

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Proposition 9.19

Proposition 9.19. If a metric space X is sequentially compact, then it is complete and totally bounded.

Proof. Let metric space X be sequentially compact. ASSUME X is not totally bounded. Then for some $\varepsilon > 0$ there is not cover of X by a finite number of open balls of radius ε. Select a point $x_1 \in X$. Since X is not contained in $B(x_1, \varepsilon)$, we may choose $x_2 \in X$ such that $\rho(x_1, x_2) \geq \varepsilon$. Now since X is not contained in $B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$, we may choose $x_2 \in X$ for which $\rho(x_3, x_2) \geq \varepsilon$ and $\rho(x_3, x_1) \geq \varepsilon$. In this way we obtain a sequence $\{x_n\}$ in X with the property that $\rho(x_n, x_k) \geq \varepsilon$ for $n \neq k$.

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Proposition 9.19 (continued)

Proposition 9.19. If a metric space X is sequentially compact, then it is complete and totally bounded.

Proof (continued). Again, let metric space X be sequentially compact. To show that X is complete, suppose $\{x_n\}$ is a Cauchy sequence in X. Since X is sequentially compact, a subsequence of $\{x_n\}$ converges to some point $x \in X$. A Cauchy sequence with a convergent subsequence is convergent (by Problem 9.38), so Cauchy sequence $\{x_n\}$ converges and X is complete, as claimed. □

Proposition 9.21

Proposition 9.21. Let f be a continuous mapping from a compact metric space X to a metric space Y. Then its image $f(X)$ is also compact.

Proof. Let $\{\mathcal{O}_n\}_{\lambda\in\Lambda}$ be an open covering of $f(X)$. Since f is continuous, Proposition 9.8 implies that each $f^{-1}(\mathcal{O}_\lambda)$ is open, so that $\{f^{-1}(\mathcal{O}_\lambda)\}_{\lambda\in\Lambda}$ is an open cover of X. Since X is compact by hypothesis, there is a finite subcollection $\{f^{-1}(\mathcal{O}_{\lambda_1}), f^{-1}(\mathcal{O}_{\lambda_2}), \ldots, f^{-1}(\mathcal{O}_{\lambda_n})\}$ that also covers X. Since f maps X onto $f(X)$, the finite collection $\{\mathcal{O}_{\lambda_1}, \mathcal{O}_{\lambda_2}, \ldots, \mathcal{O}_{\lambda_n}\}$ covers $f(X)$. Since $\{\mathcal{O}_n\}_{\lambda\in\Lambda}$ is an arbitrary cover of $f(X)$, then $f(X)$ is compact, as claimed. □

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Proposition 9.22. Extreme Value Theorem

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Let X be a metric space. Then X is compact if and only if every continuous real-valued function on X takes a maximum and a minimum value.

Proof. First, suppose X is compact. Let the function $f : X \to \mathbb{R}$ be continuous. By Proposition 9.21, $f(X)$ is a compact set of real numbers. By Theorem 9.20 ((ii) implies (i)), $f(X)$ is closed and bounded. Since $\mathbb{R}$ is complete (so set $f(X)$ with upper and lower bounds has a lub and glb) and $f(X)$ is closed (it contains is lub and glb), the f has a maximum value (namely the lub of $f(X)$) and a minimum value (namely the glb of $f(X)$).

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Second, suppose every continuous real-valued function on X takes on a maximum and minimum value. By Theorem 9.17, to show that X is compact it is sufficient to show that X is totally bounded and complete. We argue by contradiction to show that X is totally bounded.

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Proposition 9.22. Extreme Value Theorem (continued 1)

Proof (continued). ASSUME that X is not totally bounded. As shown in the proof of Proposition 9.19 (the first half where X is assumed to be not totally bounded), there is some $r > 0$ and sequence $\{x_n\}_{n=1}^{\infty}$ in X such that the collection of open balls $\{B(x_n, t)\}_{n=1}^{\infty}$ is disjoint. For each $n \in \mathbb{N}$, define the function $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} r/2 - \rho(x, x_n) & \text{if } \rho(x, x_n) \leq r/2 \\ 0 & \text{otherwise.} \end{cases}$$

The define the function $f : X \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} n f_n(x) \text{ for all } x \in X.$$

Since each f_n is continuous and vanishes outside $B(x_n, r/2)$ and the collection $\{B(x_n, r)\}_{n=1}^{\infty}$ is disjoint, then f is "properly defined" (or "well-defined") and continuous.

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Proposition 9.22. Extreme Value Theorem (continued 2)

Proof (continued). Since $r > 0$ is fixed and $f(x_n) = nr/2$ for each $n \in \mathbb{N}$, then f is unbounded above and therefore does not take on a maximum value. But this is a CONTRADICTION to the fact that f takes on a maximum and minimum value. So the assumption that X is not totally bounded is false. Hence X is totally bounded.

Now we show that X is complete. Let $\{x_n\}$ be a Cauchy sequence in X. Let $\varepsilon > 0$. Then for some $N \in \mathbb{N}$ we have for all $m, n \geq N$ that $\rho(x_n, x_m) < \varepsilon$. Then for each $x \in X$ and for $m, n \geq N$ we have $\rho(x, x_n) \leq \rho(x, x_m) + \rho(x_m, x_n)$ and $\rho(x, x_m) \leq \rho(x, x_n) + \rho(x_n, x_m)$, and for $m, n \geq N$ we have $\rho(x, x_n) - \rho(x, x_m) \leq \rho(x_m, x_n) \leq \varepsilon$ and $\rho(x, x_m) - \rho(x, x_n) \leq \rho(x_n, x_m) \leq \varepsilon$. That is, for $m, n \geq N$ we have $|\rho(x, x_n) - \rho(x, x_m)| \leq \varepsilon$ so that $\{\rho(x, x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers for every $x \in X$. Since $\mathbb{R}$ is complete, then $\{\rho(x, x_n)\}_{n=1}^{\infty}$ converges to some real number. Define $g : X \to \mathbb{R}$ by $g(x) = \lim_{n\to\infty} \rho(x, x_n)$ for all $x \in X$. Notice g is nonnegative.

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Proposition 9.22. Extreme Value Theorem.
Let X be a metric space. Then X is compact if and only if every continuous real-valued function on X takes a maximum and a minimum value.

Proof (continued). We now show that g is continuous. Let $\varepsilon > 0$ and $x \in X$. Consider arbitrary $y \in X$ with $\rho(x, y) < \delta = \varepsilon$. Now $\lim_{n\to\infty} \rho(x, x_n) = f(x)$ and $\lim_{n\to\infty} \rho(y, x_n) = f(y)$. By the triangle inequality, for all $x, y, x_n \in X$ we have $\rho(x, x_n) \le \rho(x, y) + \rho(y, x_n)$, or $\rho(x, x_n) - \rho(y, x_n) \le \rho(x, y)$. Then with $\rho(x, y) < \delta = \varepsilon$ we have

$$|g(x) - g(y)| = |\lim_{n\to\infty} \rho(x, x_n) - \lim_{n\to\infty} \rho(y, x_n)|$$

$$= |\lim_{n\to\infty}(\rho(x, x_n) - \rho(y, x_n))| \le |\lim_{n\to\infty} \rho(x, y)| = \rho(x, y) < \varepsilon.$$

Therefore g is continuous at x and, since x is an arbitrary element of X, then g is continuous on X. By hypothesis, there is $z \in X$ at which g takes on a minimum value.

Proposition 9.22. Extreme Value Theorem (continued 4)

Proposition 9.22. Extreme Value Theorem.
Let X be a metric space. Then X is compact if and only if every continuous real-valued function on X takes a maximum and a minimum value.

Proof (continued). Since $\{x_n\}$ is Cauchy, then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \geq N$ we have $\rho(x_m, x_n) < \varepsilon$. Therefore with $k \in \mathbb{N}$ and $\varepsilon = 1/k$, there is some x_{n_k} in $\{x_n\}$ such that $g(x_{n_k}) = \lim_{n\to\infty} \rho(x_{n_k}, x_n) < 1/k$. Since g is nonnegative, this implies that the infimum of function values is 0. So the minimum of g must be 0 and $g(z) = 0$. Then $g(z) = \lim_{n\to\infty} \rho(z, x_n) = 0$; that is, $\lim_{n\to\infty} x_n = z$ so that $\{x_n\}$ converges. Since $\{x_n\}$ is an arbitrary Cauchy sequence, then X is complete, as claimed. □

Note. Function g is defined in terms of Cauchy sequence $\{x_n\}$, and this determines point z. That is, point z is dependent on the Cauchy sequence, as we would expect.

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The Lebesgue Covering Lemma

The Lebesgue Covering Lemma. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of a compact metric space X. Then there is a number $\varepsilon > 0$, such that for each $x \in X$, the open ball $B(x, \varepsilon)$ is contained in some member of the cover.

Proof. ASSUME there is no such positive Lebesgue number. Then for each $n \in \mathbb{N}$, $1/n$ fails to be a Lebesgue number. That is, there is a ball $B(x_n, 1/n)$, centered at some point x_n, which fails to be contained in any member of the cover. Consider the resulting sequence $\{x_n\}$. Since X is hypothesized to be compact, then it is sequentially compact by the Characterization of Compactness for a Metric Space (Theorem 9.16; the (ii) implies (iii) part). Hence there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to some point $x_0 \in X$. There is some index $\lambda_0 \in \Lambda$ for which $x_0\ in \mathcal{O}_{\lambda_0}$. Since $\mathcal{O}_{\lambda_0}$ is open, then there is a ball centered at x_0, $B(x_0, r_0)$, for which $B(x_0, r_0) \subseteq \mathcal{O}_{\lambda_0}$.

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The Lebesgue Covering Lemma (continued)

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Proof (continued). Since $\{x_{n_k}\}$ converges to x_0, the we may choose k for which $\rho(x_0, x_{n_k}) < r_0/2$ and $1/n_k < r_0/2$. For $x \in B(x_{n_k}, 1/n_k)$ we have $\rho(x, x_{n_k}) < 1/n_k$ so that, by the Triangle Inequality,

$$\rho(x, x_0) \leq \rho(x, x_{n_k}) + \rho(x_{n_k}, x_0) < 1/n_k + r_0/2 < r_0/2 + r_0/2 = r_0.$$

That is, $B(x_{n_k}, 1/n_k) \subseteq \mathcal{O}_{\lambda_0}$. But this CONTRADICTS the choice of x_{n_k} as a point for which $B(x_{n_k}, 1/n_k)$ fails to be contained in some member of the cover. Therefore, the assumption that there is no such positive Lebesgue number is false, and so there is Lebesgue number $\varepsilon > 0$ as claimed. □

Proposition 9.23

Proposition 9.23. A continuous mapping from a compact metric space (X, ρ) into a metric space (Y, σ) is uniformly continuous.

Proof. Let f be a continuous mapping from X to Y. Let $\varepsilon > 0$. By the ε/δ Criterion for Continuity (Theorem 9.3.A), for each $x \in X$ there is $\delta_x > 0$ for which if $\rho(x, x') < \delta_x$ then $\sigma(f(x), f(x')) < \varepsilon/2$. With $\mathcal{O}_x = B(x, \delta_x)$ we have (by the triangle inequality for metric σ):

$$\sigma)f(u), f(v)) \leq \sigma(f(u), f(x)) + \sigma(f(x), f(v)) < \varepsilon \text{ if } u, v \in \mathcal{O}_x. \qquad (5)$$

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Since (X, ρ) is compact, by the Lebesgue Covering Lemma the open cover $\{\mathcal{O}_n\}_{x \in X}$ has a Lebesgue number, say δ. Then for $u, v \in X$, if $\rho(u, v) < \delta$ then there is some $x \in X$ for which $u \in B(v, \delta) \subseteq \mathcal{O}_x$. Therefore, by (5), $\sigma(f(u), f(v)) < \varepsilon$; that is, f is uniformly continuous on X. □

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