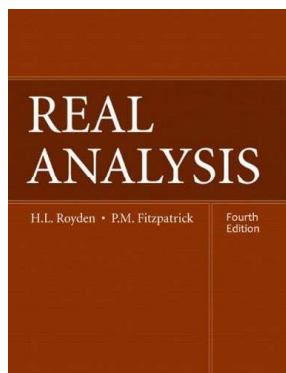


## Real Analysis

### Chapter 9. Metric Spaces: General Properties

#### 9.6. Separable Metric Spaces—Proofs of Theorems



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Proposition 9.24

### Proposition 9.24

**Proposition 9.24.** A compact metric space is separable.

**Proof.** Let  $X$  be a compact metric space. By Propositions 9.18 and 9.19,  $X$  is totally bounded. By the definition of “totally bounded,” for each  $n \in \mathbb{N}$  space  $X$  can be covered by a finite number of balls of radius  $1/n$ . Consider the (countable) collection of all such balls for  $n \in \mathbb{N}$ . Let  $D$  be the collection of points that are centers of one of these balls. Then  $D$  is countable and (since for any  $\varepsilon > 0$  we have  $\varepsilon < 1/n$  for some  $n \in \mathbb{N}$ ) dense in  $X$ . Therefore, by the definition of separable,  $X$  is a separable metric space, as claimed.  $\square$

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Proposition 9.25

### Proposition 9.25

**Proposition 9.25.** A metric space  $X$  is separable if and only if there is a countable collection  $\{\mathcal{O}_n\}_{n=1}^{\infty}$  of open subsets of  $X$  such that any open subset of  $X$  is the union of a subcollection of  $\{\mathcal{O}_n\}_{n=1}^{\infty}$ .

**Proof.** First, suppose  $X$  is separable. Let  $D$  be a countable dense subset of  $X$ . If  $D$  is finite then by its denseness in  $X$  we must have  $X = D$ . So without loss of generality, we can assume that  $D$  is countably infinite. Let  $\{x_n\}$  be an enumeration of  $D$ . Then  $\{B(x_n, 1/m)\}_{n,m \in \mathbb{N}}$  is a countable collection of open subsets of  $X$ . Let  $\mathcal{O}$  be an open subset of  $X$ . Let  $x \in \mathcal{O}$ . Since  $\mathcal{O}$  is open, then there is  $m \in \mathbb{N}$  for which  $B(x, 1/m) \subset \mathcal{O}$ . Now  $x$  is a point of closure of  $D$  by Note 9.6.A, so there is  $n \in \mathbb{N}$  for which  $x_n \in D \cap B(x, 1/(2m))$ . Then with  $n, m \in \mathbb{N}$  satisfying these conditions,  $x \in B(x_n, 1/(2m))$  and  $x \in B(x_n, 1/(2m)) \subseteq B(x, 1/m) \subset \mathcal{O}$ . Since  $x$  is an arbitrary element of  $\mathcal{O}$ , we have that open set  $\mathcal{O}$  is the union of some subcollection of  $\{B(x_n, 1/m)\}_{n,m \in \mathbb{N}}$ , as claimed.

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Proposition 9.25

### Proposition 9.25 (continued)

**Proposition 9.25.** A metric space  $X$  is separable if and only if there is a countable collection  $\{\mathcal{O}_n\}_{n=1}^{\infty}$  of open subsets of  $X$  such that any open subset of  $X$  is the union of a subcollection of  $\{\mathcal{O}_n\}_{n=1}^{\infty}$ .

**Proof (continued).** Second, suppose there is a countable collection  $\{\mathcal{O}_n\}_{n=1}^{\infty}$  of open sets such that any open subsets of  $X$  is the union of a subcollection of  $\{\mathcal{O}_n\}_{n=1}^{\infty}$ . For each  $n \in \mathbb{N}$ , choose a point in  $\mathcal{O}_n$  and label it  $x_n$ . Then the set  $\{x_n\}_{n=1}^{\infty}$  is countable and is dense since every nonempty open subset of  $X$  is the union of some subcollection of  $\{\mathcal{O}_n\}_{n=1}^{\infty}$  and therefore contains points in  $\{x_n\}_{n=1}^{\infty}$ . Hence  $X$  is separable, as claimed.  $\square$

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## Proposition 9.26

**Proposition 9.26.** Every subspace of a separable metric space is separable.

**Proof.** Let  $E$  be a subspace of separable metric space  $X$ . Since  $X$  is separable, by Proposition 9.25 there is a countable collection  $\{\mathcal{O}_n\}_{n=1}^{\infty}$  of open sets in  $X$  such that each open set in  $X$  is a union of some subcollection of  $\{\mathcal{O}_n\}_{n=1}^{\infty}$ . Thus  $\{\mathcal{O}_n \cap E\}_{n=1}^{\infty}$  is a countable collection of subsets of  $E$ , each of which is open by Proposition 9.2. Since each open subset of  $E$  is the intersection of  $E$  with an open subset of  $X$ , every open subset of  $E$  is a union of a subcollection of  $\{\mathcal{O}_n \cap E\}_{n=1}^{\infty}$ . Then by Proposition 9.25,  $E$  is separable, as claimed.  $\square$