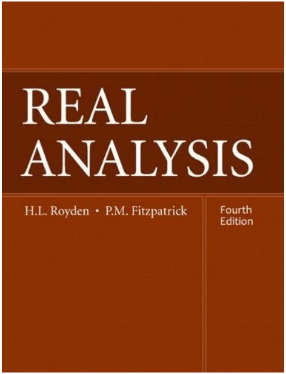
Real Analysis

Chapter 9. Metric Spaces: General Properties

9.6. Separable Metric Spaces—Proofs of Theorems



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Proposition 9.24

Proposition 9.24. A compact metric space is separable.

Proof. Let X be a compact metric space. By Propositions 9.18 and 9.19, X is totally bounded. By the definition of "totally bounded," for each $n \in \mathbb{N}$ space X can be covered by a finite number of balls of radius $1/n$. Consider the (countable) collection of all such balls for $n \in \mathbb{N}$. Let D be the collection of points that are centers of one of these balls. Then D is countable and (since for any $\varepsilon > 0$ we have $\varepsilon < 1/n$ for some $n \in \mathbb{N}$) dense in X. Therefore, by the definition of separable, X is a separable metric space, as claimed. □

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Proposition 9.25

Proposition 9.25. A metric space X is separable if and only if there is a countable collection $\{\mathcal{O}_n\}_{n=1}^{\infty}$ of open subsets of X such that any open subset of X is the union of a subcollection of $\{\mathcal{O}_n\}_{n=1}^{\infty}$.

Proof. First, suppose X is separable. Let D be a countable dense subset of X. If D is finite then by its denseness in X we must have $X = D$. So without loss of generality, we can assume that D is countably infinite. Let $\{x_n\}$ be an enumeration of D. Then $\{B(x_n, 1/m)\}_{n,m\in\mathbb{N}}$ is a countable collection of open subsets of X.

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Proof (continued). Second, suppose there is a countable collection $\{\mathcal{O}_n\}_{n=1}^{\infty}$ of open sets such that any open subsets of X is the union of a subcollection of $\{\mathcal{O}_n\}_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, choose a point in $\mathcal{O}_n$ and label it x_n. Then the set $\{x_n\}_{n=1}^{\infty}$ is countable and is dense since every nonempty open subset of X is the union of some subcollection of $\{\mathcal{O}_n\}_{n=1}^{\infty}$ and therefore contains points in $\{x_n\}_{n=1}^{\infty}$. Hence X is separable, as claimed. □

Proposition 9.26

Proposition 9.26. Every subspace of a separable metric space is separable.

Proof. Let E be a subspace of separable metric space X. Since X is separable, by Proposition 9.25 there is a countable collection $\{\mathcal{O}_n\}_{n=1}^{\infty}$ of open sets in X such that each open set in X is a union of some subcollection of $\{\mathcal{O}_n\}_{n=1}^{\infty}$. Thus $\{\mathcal{O}_n \cap E\}_{n=1}^{\infty}$ is a countable collection of subsets of E, each of which is open by Proposition 9.2. Since each open subset of E is the intersection of E with an open subset of X, every open subset of E is a union of a subcollection of $\{\mathcal{O}_n \cap E\}_{n=1}^{\infty}$. Then by Proposition 9.25, E is seprable, as claimed. □

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