Real Analysis 1, MATH 5210, Fall 2022

Homework 4, Section 2.2 Lebesgue Outer Measure, 2.3 The σ-Algebra of Lebesgue Measurable Sets

Due Saturday, September 17, at 11:59 p.m.

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

2.7. Prove that for any bounded set E, there is a G_δ set G for which $E \subset G$ and $m^*(G) = m^*(E)$. Set G is called the *measurable cover* of E. See Theorem 3.1 of the supplemental notes to Section 2.3. (In fact, this result also holds if set E is not bounded, as long as it is of finite measure.)

2.2.B. Prove that if we define the outer measure of a set $E \subset \mathbb{R}$ as

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} m^*(G_n) \,\middle|\, E \subset \cup_{n=1}^{\infty} G_n \text{ and each } G_n \text{ is a bounded open set} \right\},$$

then for all $E \subset \mathbb{R}$ we have $\lambda^*(E) = m^*(E)$. NOTE: This shows that we could use bounded open *sets* to define outer measure instead of bounded open *intervals* (some texts use this approach to outer measure; for example, A.M. Bruckner, J.B. Bruckner, and B.S. Thomson's *Real Analysis*, Prentice Hall (1997)). HINT: Show

$$\left\{ \sum_{n=1}^{\infty} \ell(I_n) \,\middle|\, E \subset \cup_{n=1}^{\infty} I_n \text{ and each } I_n \text{ is a bounded open interval} \right\}$$

$$= \left\{ \sum_{n=1}^{\infty} m^*(G_n) \,\middle|\, E \subset \cup_{n=1}^{\infty} G_n \text{ and each } G_n \text{ is a bounded open set} \right\}.$$

You may assume Problem 2.2.A: If G is a bounded open set, then $m^*(G)$ equals the sum of the lengths of its constituent countable disjoint open intervals.

2.14. Prove that if a set E has finite positive outer measure, then there is a bounded subset of E that also has positive outer measure. HINT: Consider the contrapositive.